Consumption-Based Asset Pricing in Continuous Time

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Introduction

The Optimal Portfolio and Consumption in Continuous Time notebook solved the investor's problem in partial equilibrium: prices were given, and the SDF $\Lambda_t = e^{-\delta t}u'(c_t^*)$ was the investor's own discounted marginal utility. This notebook turns the question around. We assume a representative investor, so that c is aggregate consumption, and ask what prices must be for the investor to be willing to consume it. Consumption is now the primitive and prices are the output.

This is the setting in which the SDF has empirical content, since aggregate consumption can be measured. It delivers Breeden's consumption CAPM, a formula for the real interest rate, and, in the simplest case of a Lucas tree with constant expected growth, closed-form expressions for asset prices. It also delivers the equity premium and risk-free rate puzzles, which motivate the recursive utility notebook. The discrete-time counterpart is Lognormal Consumption Growth.

The Consumption-Based SDF

The consumption FOC of the previous notebook gives a second way to write the SDF. Since $\Lambda_t = e^{-\delta t}u'(c_t)$, the dynamics of Λ can be computed from consumption alone. Applying Ito's lemma for a general felicity function,

$$
\begin{aligned}
d\Lambda &= \frac{\partial \Lambda}{\partial c}dc + \frac{1}{2}\frac{\partial^2 \Lambda}{\partial c^2}(dc)^2 + \frac{\partial \Lambda}{\partial t}dt \\
&= e^{-\delta t}u''(c)dc + \frac{1}{2}e^{-\delta t}u'''(c)(dc)^2 - \delta e^{-\delta t}u'(c)dt,
\end{aligned}
$$

or

$$\frac{d\Lambda}{\Lambda} = -\delta dt + \frac{1}{2}\frac{c^2 u'''(c)}{u'(c)}\left(\frac{dc}{c}\right)^2 + \frac{cu''(c)}{u'(c)}\frac{dc}{c}.$$

For power utility, $u(c) = c^{1-\gamma}/(1-\gamma)$, we have $cu''/u' = -\gamma$ and $c^2u'''/u' = \gamma(\gamma+1)$, so

$$\frac{d\Lambda}{\Lambda} = -\delta dt + \frac{1}{2}\gamma(\gamma+1)\left(\frac{dc}{c}\right)^2 - \gamma\frac{dc}{c}.$$

Write consumption growth as

$$\frac{dc}{c} = \mu_c dt + \sigma_c dB_c,$$

where μ_c and σ_c may depend on the state of the economy. Substituting yields

$$\frac{d\Lambda}{\Lambda} = \left(-\delta + \frac{1}{2}\gamma(\gamma+1)\sigma_c^2 - \gamma\mu_c\right)dt - \gamma\sigma_c dB_c. \tag{1}$$

The Consumption CAPM

By Discount Factors in Continuous Time, any asset S paying a dividend yield D/S satisfies the fundamental pricing equation

$$\mathsf{E}\left(\frac{dS}{S}\right) + \frac{D}{S}\,dt - r\,dt = -\frac{d\Lambda}{\Lambda}\,\frac{dS}{S}. \tag{2}$$

With the SDF (1), only the dB_c term contributes to the covariance, and

$$\mathsf{E}\left(\frac{dS}{S}\right) + \frac{D}{S}\,dt - r\,dt = \gamma\,\frac{dc}{c}\,\frac{dS}{S}. \tag{3}$$

This is **Breeden's consumption CAPM** (Breeden 1979): the risk premium of any asset is risk aversion times the covariance of its return with consumption growth.

The CCAPM is the same pricing relation as Merton's ICAPM from the previous notebook, written in different variables. There, the $d\mathbf{B}$ part of $d\Lambda/\Lambda$ was computed from $V_W(W, \mathbf{z})$ and had two pieces, one for wealth and one for the state variables. Here it is computed from $u'(c)$, and since $u'(c^*) = V_W$ the two computations must agree:

$$-\gamma \left.\frac{dc}{c}\right|_{d\mathbf{B}} = -\text{rra} \left.\frac{dW}{W}\right|_{d\mathbf{B}} + \left.\frac{(\nabla_z V_W)'}{V_W} d\mathbf{z}\right|_{d\mathbf{B}}.$$

Consumption growth summarizes everything the investor cares about. When a shock lowers wealth or worsens investment opportunities, the investor cuts consumption, so a single consumption factor replaces the wealth factor and all the state-variable factors. The identity holds for any optimizing investor and that investor's own consumption. With a representative investor it applies to aggregate consumption, which is what makes it testable. Hansen and Singleton (1983) and Breeden et al. (1989) are early tests.

The Risk-Free Rate

The instantaneous risk-free rate equals minus the drift of $d\Lambda/\Lambda$ in (1):

$$r = \delta + \gamma\mu_c - \frac{1}{2}\gamma(\gamma+1)\sigma_c^2. \tag{4}$$

In terms of the expected growth of log consumption, $g = \mu_c - \frac{1}{2}\sigma_c^2$, this is

$$r = \delta + \gamma g - \frac{1}{2}\gamma^2\sigma_c^2, \tag{5}$$

the form used in the one-factor Gaussian notebook. There the formula is read in reverse, as the consumption path an investor chooses when facing a given interest rate. Here consumption is given and the formula determines the interest rate.

The expression has three components. Real interest rates are high when impatience (δ) is high, since more impatient investors demand a high return to save. They are high when expected consumption growth (μ_c) is high, since agents expecting rising consumption need to save less, pushing bond prices down. Finally, they are low when consumption volatility (σ_c) is high. This is the **precautionary savings** effect: more uncertain future consumption raises the demand for safe assets, pushing bond prices up and yields down.

A Lucas Tree with Constant Growth

The simplest complete equilibrium model is an exchange economy with a single **Lucas tree** (Lucas 1978). The tree produces a perishable dividend that is the economy's only source of consumption, so in equilibrium aggregate consumption equals the dividend, $D = c$. Consumption growth is independent over time with constant coefficients,

$$\frac{dc}{c} = \mu_c \, dt + \sigma_c \, dB,$$

so $\ln c$ grows at the constant expected rate $g = \mu_c - \frac{1}{2}\sigma_c^2$ with volatility σ_c. The tree is in unit supply and the risk-free asset in zero net supply, so the representative investor must hold the tree and nothing else. The tree is therefore both the market portfolio and the investor's wealth.

Let P denote the price of the tree. By the definition of the SDF,

$$P_t = \mathsf{E}_t\left[\int_0^\infty \frac{\Lambda_{t+\tau}}{\Lambda_t} c_{t+\tau} \, d\tau\right] = c_t \; \mathsf{E}_t\left[\int_0^\infty e^{-\delta\tau} \left(\frac{c_{t+\tau}}{c_t}\right)^{1-\gamma} d\tau\right].$$

Since $\ln(c_{t+\tau}/c_t)$ is normal with mean $g\tau$ and variance $\sigma_c^2\tau$, the lognormal moment formula gives $\mathsf{E}_t\left[(c_{t+\tau}/c_t)^{1-\gamma}\right] = \exp\left[\left((1-\gamma)g + \frac{1}{2}(1-\gamma)^2\sigma_c^2\right)\tau\right]$. Integrating over τ:

Property 1 (Lucas Tree with Constant Growth)**.** *The price-dividend ratio of the tree is constant,*

$$\frac{P}{c} = \frac{1}{\delta - (1-\gamma)g - \frac{1}{2}(1-\gamma)^2\sigma_c^2}, \tag{6}$$

provided the denominator is positive. The dividend yield is

$$q = \frac{c}{P} = \delta - (1-\gamma)g - \frac{1}{2}(1-\gamma)^2\sigma_c^2,$$

and the risk premium on the tree is

$$\mu_P + q - r = \gamma\sigma_c^2. \tag{7}$$

Because P/c is constant, the price inherits the dynamics of consumption: $dP/P = dc/c$, so

$\mu_P = \mu_c$ and the tree has volatility σ_c and correlation one with consumption. The CCAPM (3) then gives the premium (7) directly. It can also be checked by substituting q and (4) into $\mu_c + q - r$: all terms cancel except $\gamma\sigma_c^2$.

Three features of the solution are worth noting.

1. **Market clearing.** Merton's rule from the previous notebook, with a constant opportunity set, gives the representative investor a risky share of $(\mu_P + q - r)/(\gamma\sigma_c^2) = 1$. The investor is content to hold exactly the tree and no bonds, so the equilibrium is consistent with the investor's optimal portfolio.

2. **Growth and valuation.** A rise in expected growth g changes the dividend yield by $\partial q/\partial g = \gamma - 1$. With log utility, $q = \delta$ and valuations do not respond to growth at all. For $\gamma > 1$, higher growth *raises* the dividend yield and lowers the price-dividend ratio: better cash-flow prospects are more than offset by the higher interest rate in (5), because the elasticity of intertemporal substitution $1/\gamma$ is below one. This is the same discount-rate effect that makes wealth fall on good news in the one-factor Gaussian notebook.

3. **Sharpe ratio.** The tree's Sharpe ratio is $\gamma\sigma_c^2/\sigma_c = \gamma\sigma_c$, the volatility of the SDF. Since the tree is perfectly correlated with consumption, it attains the Hansen-Jagannathan bound of Discount Factors in Continuous Time.

Example 1. Let $\delta = 0.01$, $g = 0.02$, $\sigma_c = 0.01$ and $\gamma = 2$. The risk-free rate is

$$r = 0.01 + 2 \times 0.02 - \tfrac{1}{2}(4)(0.0001) = 4.98\%,$$

the dividend yield is

$$q = 0.01 + 0.02 - \tfrac{1}{2}(0.0001) \approx 3.0\%,$$

so $P/c \approx 33.4$, and the equity premium is $\gamma\sigma_c^2 = 2 \times 0.0001 = 0.02\%$. The tree has a volatility of 1% and a Sharpe ratio of 0.02. □

In U.S. data (Campbell 2003), the equity premium is around 6%, stock market volatility around 16%, the Sharpe ratio around 0.4 to 0.5, and the real interest rate around 1%. The model misses on all four counts: the premium is too small by a factor of three hundred, prices are far too smooth, and the interest rate is too high.

The volatility gap is sometimes called the **volatility puzzle**: in this model stock prices move one-for-one with consumption, which is far smoother than the stock market (Shiller 1981). Fixing it requires a price-dividend ratio that moves over time, as in the one-factor model where P/c depends on the state x. The premium and interest-rate gaps are the subject of the next section.

The Equity Premium and Risk-Free Rate Puzzles

The Lucas tree is perfectly correlated with consumption. For a general asset with

$$\frac{dS}{S} = \mu_S dt + \sigma_S dB_S$$

such that $(dB_S)(dB_c) = \rho\, dt$, the CCAPM (3) gives

$$\mu_S + D/S - r = \gamma\rho\sigma_c\sigma_S.$$

The risk premium of any risky asset is higher when risk aversion (γ) is high and when the covariance of its return with consumption growth is high.

Since $|\rho| \leq 1$, the previous expression implies

$$\left|\frac{\mu_S + D/S - r}{\sigma_S}\right| \leq \gamma\sigma_c.$$

This is the Hansen-Jagannathan bound (Hansen and Jagannathan 1991) of the SDF notebook, with SDF volatility $||\boldsymbol{\lambda}|| = \gamma\sigma_c$: no asset can have a Sharpe ratio larger than the volatility of the consumption-based SDF.

Example 2. The Sharpe ratio of the U.S. stock market is around 0.5, while the volatility of aggregate consumption growth is around $\sigma_c = 0.01$. Even with $\rho = 1$, the bound requires $\gamma \geq 0.5/0.01 = 50$. The measured correlation between stock returns and consumption growth is closer to $\rho = 0.25$, which raises the requirement to

$$\gamma \geq \frac{0.5}{0.25 \times 0.01} = 200.$$

Such values also break the risk-free rate. With $\delta = 0.01$, $\mu_c = 0.02$ and $\gamma = 50$, (4) gives

$$r = 0.01 + 50 \times 0.02 - \tfrac{1}{2}(50)(51)(0.01)^2 \approx 0.88,$$

an 88% real interest rate, against historical real rates of about 1%. $\square$

These are the equity premium puzzle of Mehra and Prescott (1985) and the risk-free rate puzzle of Weil (1989). The two puzzles are linked because a single parameter does two jobs. Under power utility, γ is both the coefficient of relative risk aversion, which governs the equity premium, and the inverse of the elasticity of intertemporal substitution, which governs how strongly the interest rate responds to expected growth. Raising γ to match the premium makes the investor extremely unwilling to substitute consumption over time, so with positive expected growth the interest rate explodes.

To resolve these puzzles, researchers have introduced preferences that generate a more volatile stochastic discount factor without implausible interest rates, such as recursive Epstein-Zin preferences (Epstein and Zin 1989; Weil 1990), which separate risk aversion from the elasticity of intertemporal substitution, or habits (Campbell and Cochrane 1999). The recursive utility notebook takes up the first route.

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