Discount Factors in Continuous Time

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Price Processes

We work in a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ where all randomness is generated by K independent Brownian motions $\mathbf{B} = \begin{pmatrix} B_1 & B_2 & \dots & B_K \end{pmatrix}'$ satisfying

$$(d\mathbf{B})(d\mathbf{B})' = \mathbf{I}dt,$$

where $\mathbf{I}$ is the $K \times K$ identity matrix. The K shocks capture distinct sources of uncertainty in the economy, such as technology, demand, and interest rates.

Although the Brownian motions in $\mathbf{B}$ are independent, we can always construct a correlated Brownian motion by taking a linear combination. For example, let $Z = \rho B_1 + \sqrt{1-\rho^2}B_2$ where $|\rho| \leq 1$. Then Z is a Brownian motion since $(dZ)^2 = \rho^2 dt + (1-\rho^2)dt = dt$, and it is instantaneously correlated with B_1 since $(dZ)(dB_1) = \rho(dB_1)^2 = \rho\, dt$. More generally, for any constant vector $\mathbf{a} \neq \mathbf{0}$,

$$Z = \frac{1}{\sqrt{\mathbf{a}'\mathbf{a}}}\mathbf{a}'\mathbf{B}$$

is a Brownian motion with $(dZ)(dB_k) = a_k/\sqrt{\mathbf{a}'\mathbf{a}}\, dt$. The weights may also vary over time and across states, as long as the weight vector keeps unit length, a consequence of Lévy's characterization of Brownian motion (Shreve 2004, sec. 4.6). This means that by driving different assets with overlapping linear combinations of $\mathbf{B}$, we can capture any pattern of co-movement among returns.

Example 1. Suppose $K = 3$. To build a Brownian motion with correlation $\rho = 0.6$ with B_1, set $Z = 0.6B_1 + 0.8B_2$, since $\sqrt{1-0.6^2} = 0.8$. Then $(dZ)^2 = (0.36 + 0.64)\, dt = dt$ and $(dZ)(dB_1) = 0.6\, dt$.

Now take $\mathbf{a} = (1, 2, 2)'$, so $\sqrt{\mathbf{a}'\mathbf{a}} = \sqrt{1 + 4 + 4} = 3$ and

$$Z_1 = \tfrac{1}{3}B_1 + \tfrac{2}{3}B_2 + \tfrac{2}{3}B_3.$$

Its correlations with the three shocks are $1/3$, $2/3$ and $2/3$, and $(dZ_1)^2 = (1+4+4)/9\, dt = dt$. Consider two more combinations with unit-length weights,

$$Z_2 = \tfrac{2}{3}B_1 + \tfrac{2}{3}B_2 + \tfrac{1}{3}B_3, \qquad Z_3 = \tfrac{2}{3}B_1 - \tfrac{2}{3}B_2 + \tfrac{1}{3}B_3.$$

Since $(dB_j)(dB_k) = 0$ for $j \neq k$, the correlation between two such combinations is the dot product of their weight vectors:

$$(dZ_1)(dZ_2) = \frac{1 \cdot 2 + 2 \cdot 2 + 2 \cdot 1}{9}\, dt = \frac{8}{9}\, dt, \qquad (dZ_1)(dZ_3) = \frac{1 \cdot 2 - 2 \cdot 2 + 2 \cdot 1}{9}\, dt = 0.$$

Z_1 and Z_2 load on the shocks in similar proportions and are highly correlated. Z_1 and Z_3 are uncorrelated even though both depend on all three shocks, because the positive and negative overlaps cancel. □

We model the price of a risky asset as a diffusion

$$\frac{dS}{S} = \mu_S(\cdot)\, dt + \sigma_S(\cdot)\, dB_S,$$

where the drift $\mu_S(\cdot)$ and return volatility $\sigma_S(\cdot)$ may depend on time t, the state $\omega \in \Omega$, or other state variables, and B_S is a Brownian motion given by some linear combination of $\mathbf{B}$.

We also assume a risk-free money-market account β that earns the continuously-compounded rate r. Starting from β_0,

$$\frac{d\beta}{\beta} = r\, dt, \tag{1}$$

which solves to

$$\beta_t = \beta_0 \exp\left(\int_0^t r_s\, ds\right). \tag{2}$$

The rate r need not be constant; in term-structure models it follows its own diffusion

$$dr = \mu_r(\cdot)\, dt + \sigma_r(\cdot)\, dB_r,$$

in which case β_t depends on the entire path of rates from 0 to t.

The Dividend-Reinvested Price

The total instantaneous return of an asset is given by

$$\frac{dS + D\,dt}{S} = \frac{dS}{S} + \frac{D}{S}dt,$$

where D/S is the **dividend yield**. Rather than modeling dividends as a separate cash payment, we model them as generating new shares at rate D/S per unit time, as if every dividend is immediately reinvested in the stock. This allows us to track total wealth without separately accounting for dividend payments. To see this, let

$$\frac{dX}{X} = \frac{D}{S}dt. \tag{3}$$

Here X represents the number of shares held by an investor who reinvests every dividend, starting from X_0 shares. Note that

$$X_t = X_0 \exp\left(\int_0^t \frac{D_u}{S_u}du\right), \tag{4}$$

so the number of shares grows at an instantaneous rate equal to the dividend yield of the asset. In other words, X_t keeps track of the total number of shares at each point in time.

The dividend-reinvested asset price is then

$$P = XS,$$

where P denotes the total value of this investment given by the number of shares times the price per share. We can find the dynamics of P by applying Ito's product rule. Since X is locally riskless (it has no Brownian $d\mathbf{B}$ term), its cross-variation with S is zero ($dX\,dS = 0$), yielding:

$$\frac{dP}{P} = \frac{dS}{S} + \frac{dX}{X} = \frac{dS}{S} + \frac{D}{S}dt.$$

The dynamics of P equal the total return on the asset: capital gains dS/S plus the dividend yield D/S.

The Pricing Equation

We seek a strictly positive process Λ, the cumulative SDF, such that ΛP is a martingale for every dividend-reinvested price P. This is the continuous-time counterpart of the one-period condition $\Lambda_t P_t = \mathrm{E}_t[\Lambda_{t+1} P_{t+1}]$. The equivalence between the absence of arbitrage and the existence of such a process goes back to Harrison and Kreps (1979) and Harrison and Pliska (1981), and Duffie (2010) and Cochrane (2009) give textbook treatments. The martingale property requires a zero conditional expected instantaneous increment:[1]

$$\mathrm{E}_t[d(\Lambda P)] = 0. \tag{5}$$

To express this in terms of the observable price S and the dividend flow, note that $dP/P = dS/S + (D/S)\,dt$, so by Ito's product rule

$$\frac{d(\Lambda P)}{\Lambda P} = \frac{d\Lambda}{\Lambda} + \frac{dP}{P} + \frac{d\Lambda}{\Lambda}\frac{dP}{P} = \frac{d(\Lambda S)}{\Lambda S} + \frac{D}{S}\,dt,$$

where the last step uses $dP/P = dS/S + (D/S)\,dt$ and the fact that the cross term $(d\Lambda/\Lambda)(D/S\,dt)$ vanishes. Thus (5) is equivalent to

$$\mathrm{E}_t[d(\Lambda S)] + \Lambda D\,dt = 0. \tag{6}$$

The money-market account β pays no dividends, so (6) gives $\mathrm{E}_t[d(\Lambda\beta)] = 0$. By Ito's product rule, $d(\Lambda\beta) = \Lambda\,d\beta + \beta\,d\Lambda$, and since $d\beta = r\beta\,dt$,

$$\mathrm{E}_t\!\left(\frac{d\Lambda}{\Lambda}\right) + r\,dt = 0 \implies \mathrm{E}_t\!\left(\frac{d\Lambda}{\Lambda}\right) = -r\,dt.$$

[1] Strictly speaking, a zero drift only makes ΛP a *local* martingale. A local martingale that is bounded below, as $\Lambda P > 0$ is, is a supermartingale, so $\Lambda_t P_t \geq \mathrm{E}_t[\Lambda_T P_T]$, and the inequality can be strict when prices contain a bubble. Ruling this out requires an integrability condition, for example that $\mathrm{E}[\sup_{s\leq T} \Lambda_s P_s] < \infty$. We assume such conditions hold throughout. Loewenstein and Willard (2000), Cox and Hobson (2005) and Heston et al. (2007) show how such bubbles arise in continuous-time models.

The drift of the SDF equals minus the risk-free rate. This holds even when r is stochastic, as long as it is adapted to the filtration.

For a risky asset S, Ito's product rule gives

$$\frac{d(\Lambda S)}{\Lambda S} = \frac{d\Lambda}{\Lambda} + \frac{dS}{S} + \frac{d\Lambda}{\Lambda}\frac{dS}{S}.$$

Taking conditional expectations and using $\mathsf{E}_t[d(\Lambda S)]/(\Lambda S) = -(D/S)\,dt$ from (6) and $\mathsf{E}_t(d\Lambda/\Lambda) = -r\,dt$:

$$\mathsf{E}_t\left(\frac{dS}{S}\right) + \frac{D}{S}\,dt = r\,dt - \frac{d\Lambda}{\Lambda}\frac{dS}{S}.$$

The cross-term $(d\Lambda/\Lambda)(dS/S)$ is the instantaneous quadratic covariation. Only the Brownian parts of the two returns contribute to it, and since $(d\mathbf{B})(d\mathbf{B})' = \mathbf{I}\,dt$ it is of order dt with a coefficient that is known at time t, so it passes through the conditional expectation unchanged. Moving $r\,dt$ to the left, the expected total return in excess of the risk-free rate equals minus the instantaneous covariance between the growth of the SDF and the asset return. This gives the fundamental pricing equation:

Property 1. *Consider an asset S that follows a diffusion*

$$\frac{dS}{S} = \mu\,dt + \sigma\,dB.$$

If the asset pays a dividend yield $q = D/S$, and there are no arbitrage opportunities, it must be the case that

$$(\mu + q - r)\,dt = -\left(\frac{d\Lambda}{\Lambda}\right)\left(\frac{dS}{S}\right). \tag{7}$$

In words, the risk premium of the asset equals minus the covariance between the growth of the SDF and the asset's return.

Example 2. Let $K = 2, r = 3\%$, and suppose the SDF loads only on the first shock:

$$\frac{d\Lambda}{\Lambda} = -0.03\,dt - 0.4\,dB_1.$$

A stock has volatility $\sigma = 20\%$, pays a dividend yield $q = 2\%$, and is driven by the Brownian motion $Z = 0.6B_1 + 0.8B_2$ of Example 1:

$$\frac{dS}{S} = \mu\, dt + 0.2\,(0.6\, dB_1 + 0.8\, dB_2)\,.$$

Only the dB_1 terms survive in the cross product, so

$$\left(\frac{d\Lambda}{\Lambda}\right)\left(\frac{dS}{S}\right) = (-0.4)(0.2)(0.6)\, dt = -0.048\, dt.$$

By (7), the risk premium is $\mu + q - r = 4.8\%$. The expected total return is $\mu + q = 3\% + 4.8\% = 7.8\%$, of which 2% comes from dividends, so the expected capital gain is $\mu = 5.8\%$.

Because the SDF loads only on B_1, only that shock is priced, and B_2 is an unpriced source of risk. The stock's exposure to B_2, with instantaneous variance $0.16^2 = 0.0256$, adds to its risk but earns nothing. A stock driven only by B_2 would earn $\mu + q = r$ whatever its volatility, and one driven by $-0.6B_1 + 0.8B_2$ would have a risk premium of -4.8%, since it pays off when marginal utility is high. The stock's Sharpe ratio is $0.048/0.2 = 0.24$, the product of the SDF volatility 0.4 and the correlation 0.6 between the stock and the priced shock. □

The Market Price of Risk

The drift of the SDF is pinned at $-r\, dt$ by the money-market account, as we derived above. The diffusion component, however, is not in general uniquely determined by the absence of arbitrage, which only requires that the SDF prices every traded asset correctly. Writing the $d\mathbf{B}$ exposure of the SDF as $-\boldsymbol{\lambda}'$, the general form is

$$\frac{d\Lambda}{\Lambda} = -r\, dt - \boldsymbol{\lambda}' d\mathbf{B}, \tag{8}$$

where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_K)' \in \mathbb{R}^K$ is the market price of risk vector. The k-th component λ_k measures the instantaneous risk premium per unit of exposure to Brownian shock B_k: an asset with exposure σ_{ik} to B_k earns a contribution $\sigma_{ik}\lambda_k$ to its risk premium.

Suppose there are $N \leq K$ securities, none of them redundant, each driven by the common shock vector $\mathbf{B}$:

$$\frac{dS_i}{S_i} = \mu_i dt + \boldsymbol{\sigma}_i' d\mathbf{B}, \tag{9}$$

where $\boldsymbol{\sigma}_i$ is a $K \times 1$ vector of return exposures, and each security pays a continuous dividend yield q_i. Applying the fundamental pricing equation (7) to asset i and using (8) with $(d\mathbf{B})(d\mathbf{B})' = \mathbf{I}\,dt$:

$$(\mu_i + q_i - r)\,dt = -\frac{d\Lambda}{\Lambda}\frac{dS_i}{S_i} = \boldsymbol{\lambda}'(d\mathbf{B})(d\mathbf{B})'\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i'\boldsymbol{\lambda}\,dt. \tag{10}$$

Stacking all N assets, with $\boldsymbol{\sigma}$ denoting the $N \times K$ matrix whose i-th row is $\boldsymbol{\sigma}_i'$, $\mathbf{q} = (D_1/S_1, \ldots, D_N/S_N)'$, and $\boldsymbol{\iota}$ an $N \times 1$ vector of ones:

$$\boldsymbol{\mu} + \mathbf{q} - r\boldsymbol{\iota} = \boldsymbol{\sigma}\boldsymbol{\lambda}. \tag{11}$$

Each asset's risk premium equals its exposure vector $\boldsymbol{\sigma}_i$ dotted with $\boldsymbol{\lambda}$. The $N \times N$ instantaneous return covariance matrix is $\boldsymbol{\sigma}\boldsymbol{\sigma}'\,dt$. Since no asset is redundant, $\boldsymbol{\sigma}$ has full row rank N and $\boldsymbol{\sigma}\boldsymbol{\sigma}'$ is invertible.

When the number of traded assets is less than the number of underlying risk sources ($N < K$), the market is **incomplete**. In this scenario, (11) is an underdetermined system: it has N equations but K unknowns, and because $\boldsymbol{\sigma}$ has full row rank it has infinitely many solutions. Economically, this non-uniqueness arises because some risks cannot be hedged by trading the available assets.

Mathematically, any two valid solutions differ by a vector $\boldsymbol{\nu}$ that lies in the null space of $\boldsymbol{\sigma}$ (i.e., $\boldsymbol{\sigma}\boldsymbol{\nu} = \mathbf{0}$). Adding this "unpriced" risk $\boldsymbol{\nu}$ to $\boldsymbol{\lambda}$ changes the SDF but leaves the risk premiums of all N assets unchanged, since $\boldsymbol{\sigma}(\boldsymbol{\lambda} + \boldsymbol{\nu}) = \boldsymbol{\sigma}\boldsymbol{\lambda}$.

Among all these valid solutions, the **minimum-norm solution** is particularly important:

$$\boldsymbol{\lambda}_{\min} = \boldsymbol{\sigma}'\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\left(\boldsymbol{\mu} + \mathbf{q} - r\boldsymbol{\iota}\right). \tag{12}$$

This is the unique $\boldsymbol{\lambda}$ that prices all N assets and lies entirely in the row space of $\boldsymbol{\sigma}$. He and Pearson (1991) and Karatzas et al. (1991) use this family of SDFs to solve consumption and portfolio problems in incomplete markets. Because the null space and the row space are orthogonal, every admissible solution can be decomposed into orthogonal components:

$$\boldsymbol{\lambda} = \boldsymbol{\lambda}_{\text{min}} + \boldsymbol{\nu}, \qquad \boldsymbol{\nu} \in \text{Null}(\boldsymbol{\sigma}).$$

By the Pythagorean theorem, the squared norm of any valid market price of risk is:

$$||\boldsymbol{\lambda}||^2 = ||\boldsymbol{\lambda}_{\text{min}}||^2 + ||\boldsymbol{\nu}||^2.$$

Since $||\boldsymbol{\nu}||^2 > 0$ whenever $\boldsymbol{\nu} \neq \mathbf{0}$, adding any unhedged risk strictly increases the overall norm. Thus, $\boldsymbol{\lambda}_{\text{min}}$ represents the least volatile pricing kernel that successfully prices the available assets. By contrast, when markets are complete ($N = K$ and $\boldsymbol{\sigma}$ is invertible), the null space is trivial ($\boldsymbol{\nu} = \mathbf{0}$) and the SDF is uniquely determined as $\boldsymbol{\lambda} = \boldsymbol{\sigma}^{-1}(\boldsymbol{\mu} + \mathbf{q} - r\boldsymbol{\iota})$. Duffie and Huang (1985) show that continuous trading in a small number of long-lived securities, K risky assets plus the money-market account, is enough to complete the market.

Example 3. Return to the stock of Example 2, with $N = 1$ traded asset and $K = 2$ shocks. Its exposure vector is $\boldsymbol{\sigma}_1 = 0.2 \times (0.6,\ 0.8)' = (0.12,\ 0.16)'$, so $\boldsymbol{\sigma} = \boldsymbol{\sigma}_1'$ is a 1×2 matrix, and its risk premium is 4.8%, so any market price of risk must satisfy the single equation

$$0.12\,\lambda_1 + 0.16\,\lambda_2 = 0.048.$$

The SDF of Example 2, $\boldsymbol{\lambda} = (0.4,\ 0)'$, is one solution, but so is $(0,\ 0.3)'$, and so is every point on this line. Since $\boldsymbol{\sigma}\boldsymbol{\sigma}' = 0.12^2 + 0.16^2 = 0.04$, the minimum-norm solution (12) is

$$\boldsymbol{\lambda}_{\text{min}} = \begin{pmatrix} 0.12 \\ 0.16 \end{pmatrix} \frac{0.048}{0.04} = \begin{pmatrix} 0.144 \\ 0.192 \end{pmatrix},$$

with norm $||\boldsymbol{\lambda}_{\text{min}}|| = \sqrt{0.144^2 + 0.192^2} = 0.24$, exactly the stock's Sharpe ratio. The null space of $\boldsymbol{\sigma}$ is spanned by $(4,\ -3)'$, since $0.12 \times 4 - 0.16 \times 3 = 0$, and indeed

$$\begin{pmatrix} 0.4 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.144 \\ 0.192 \end{pmatrix} + 0.064 \begin{pmatrix} 4 \\ -3 \end{pmatrix},$$

so the squared norms add up: $0.4^2 = 0.16 = 0.0576 + 0.1024 = 0.24^2 + 0.064^2 \times 25.$

Someone who observes only this stock cannot tell which of these SDFs is the true one. The minimum-norm solution spreads the premium over both shocks in proportion to the stock's exposures, treating all of the stock's risk as priced, so it loads on B_2 even though the SDF of Example 2 does not. Whether B_2 is priced depends on which SDF we use, and a single traded asset cannot settle it. □

The Hansen-Jagannathan Bound

This minimum-norm SDF provides a fundamental limit on asset returns. Since the return variance of asset i is $(\boldsymbol{\sigma}_i' d\mathbf{B})^2/dt = \boldsymbol{\sigma}_i'\boldsymbol{\sigma}_i$, its instantaneous return volatility is $\|\boldsymbol{\sigma}_i\| > 0$.

From (10), the absolute risk premium of asset i is $|\mu_i + q_i - r| = |\boldsymbol{\sigma}_i'\boldsymbol{\lambda}|$. Because the unpriced risk $\boldsymbol{\nu}$ is orthogonal to the asset's exposures ($\boldsymbol{\sigma}_i'\boldsymbol{\nu} = 0$), we can write this entirely in terms of the minimum-norm solution: $\boldsymbol{\sigma}_i'\boldsymbol{\lambda} = \boldsymbol{\sigma}_i'\boldsymbol{\lambda}_{\text{min}}$.

Dividing by the volatility $\|\boldsymbol{\sigma}_i\|$ and applying the Cauchy-Schwarz inequality ($|\mathbf{a}'\mathbf{b}| \leq \|\mathbf{a}\|\|\mathbf{b}\|$) yields:

$$\left|\frac{\mu_i + q_i - r}{\|\boldsymbol{\sigma}_i\|}\right| \leq \|\boldsymbol{\lambda}_{\text{min}}\|. \tag{13}$$

This is the continuous-time counterpart of the **Hansen-Jagannathan bound** (Hansen and Jagannathan 1991). It states that the absolute value of the instantaneous Sharpe ratio of any traded asset cannot exceed the norm of the minimum-norm market price of risk, and therefore cannot exceed the norm of any valid $\boldsymbol{\lambda}$. The same argument applies to any portfolio of the N assets, since a portfolio with weights $\mathbf{w}$ has exposure vector $\boldsymbol{\sigma}'\mathbf{w}$ and risk premium $\mathbf{w}'\boldsymbol{\sigma}\boldsymbol{\lambda}$.

Equality in (13) holds only for an asset or portfolio whose diffusion vector is proportional to $\boldsymbol{\lambda}_{\text{min}}$, that is, perfectly correlated with the minimum-norm pricing kernel. A positive multiple is perfectly negatively correlated with the minimum-norm SDF and earns Sharpe ratio $\|\boldsymbol{\lambda}_{\text{min}}\|$, while a negative multiple earns $-\|\boldsymbol{\lambda}_{\text{min}}\|$. Such a portfolio always exists: $\boldsymbol{\lambda}_{\text{min}}$ lies in the row space of $\boldsymbol{\sigma}$, so $\boldsymbol{\lambda}_{\text{min}} = \boldsymbol{\sigma}'\mathbf{w}$ for some weights $\mathbf{w}$, even if no single asset has that exposure. Hence $\|\boldsymbol{\lambda}_{\text{min}}\|$ is exactly the maximum attainable Sharpe ratio, and every valid SDF has volatility $\|\boldsymbol{\lambda}\| \geq \|\boldsymbol{\lambda}_{\text{min}}\|$. In the data, the U.S. equity market has historically delivered a Sharpe ratio of roughly 0.5 per year. Therefore, to be empirically plausible, any asset pricing model must feature an SDF volatility $\|\boldsymbol{\lambda}\| \geq 0.5$. With power utility the SDF volatility is risk aversion times consumption growth volatility, and consumption growth volatility is only 1 to 2% per year, so the bound requires implausibly high risk aversion. This is the equity premium puzzle of Mehra and Prescott (1985), restated by Hansen and Jagannathan (1991).

Example 4. Add a second stock to Example 3, with exposure vector $\boldsymbol{\sigma}_2 = (0.20,\ -0.15)'$, so its volatility is $\|\boldsymbol{\sigma}_2\| = 0.25$. Under the SDF of Example 2, $\boldsymbol{\lambda} = (0.4,\ 0)'$, its risk premium is $\boldsymbol{\sigma}_2'\boldsymbol{\lambda} = 0.08$. The two assets now satisfy

$$\begin{pmatrix} 0.12 & 0.16 \\ 0.20 & -0.15 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} 0.048 \\ 0.08 \end{pmatrix}.$$

With $N = K = 2$ and an invertible $\boldsymbol{\sigma}$, the market is complete and the only solution is $\boldsymbol{\lambda} = (0.4,\ 0)'$. Adding the second stock pins down the SDF and removes the ambiguity of Example 3, so now $\boldsymbol{\lambda}_{\text{min}} = \boldsymbol{\lambda}$ and the bound is $\|\boldsymbol{\lambda}_{\text{min}}\| = 0.4$.

The individual Sharpe ratios are $0.048/0.2 = 0.24$ and $0.08/0.25 = 0.32$, both below 0.4. Neither stock attains the bound because each one loads on the unpriced shock B_2. A portfolio can cancel that exposure: with weights $w_1 = 15/31$ and $w_2 = 16/31$, the B_2 loading is $(15 \times 0.16 - 16 \times 0.15)/31 = 0$ and the B_1 loading is $(15 \times 0.12 + 16 \times 0.20)/31 = 5/31$. The portfolio's exposure is proportional to $\boldsymbol{\lambda}_{\text{min}}$, so its Sharpe ratio is

$$\frac{0.4 \times 5/31}{5/31} = 0.4,$$

exactly the bound. Both weights are positive: the stocks load on B_2 with opposite signs, so holding both long cancels the unpriced risk.

In this example the two stocks happen to be uncorrelated, since $\boldsymbol{\sigma}_1'\boldsymbol{\sigma}_2 = 0.024 - 0.024 = 0$. For uncorrelated assets the squared Sharpe ratios add, which gives the same answer: $\sqrt{0.24^2 + 0.32^2} = 0.4$. Any model whose SDF has volatility below 0.4 could not price these two stocks. □

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