Optimal Portfolio and Consumption in a One-Factor Gaussian Model

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Introduction

The Optimal Portfolio and Consumption in Continuous Time notebook derived the general HJB equation, Merton's portfolio decomposition, and the SDF under additive utility. This sequel specializes those results to the most tractable nontrivial example of a time-varying investment opportunity set: the short rate is affine in one mean-reverting Gaussian state variable, and the market price of risk is constant.

The analysis is partial equilibrium. The investor takes the price processes as given, and nothing requires the investor's consumption or wealth to match any aggregate quantity. Investors with different preferences facing the same market choose different consumption paths and different portfolios. This is the setting in which hedging demand has content: in a representative-agent equilibrium the agent simply holds the market.

The payoff from this specialization is that the policy functions can be written explicitly in terms of one-dimensional integrals. We obtain three objects:

1. the **value function**,
2. the **consumption rule**, through the wealth-consumption ratio W/C,
3. the **optimal risky portfolio**, split into myopic and hedging demands.

The structure is also a useful bridge between the general HJB notebook and the Affine Recursive Utility in Continuous Time notebook: here the solution is exact because in a complete market the investor's optimal log consumption is Gaussian, so all required conditional expectations reduce to log-normal moment formulas.

The Investment Opportunity Set

The opportunity set is driven by a state variable x that follows an Ornstein-Uhlenbeck process,

$$dx = \kappa(\bar{x} - x)\, dt + \sigma_x\, dB. \tag{1}$$

The investor can trade a money-market account earning the short rate

$$r(x) = r_0 + r_1 x \tag{2}$$

and one risky asset,

$$\frac{dS}{S} = \big(r(x) + \sigma_S \lambda\big)\, dt + \sigma_S\, dB, \tag{3}$$

where S denotes the dividend-reinvested price, so that dS/S is the total return, and the volatility $\sigma_S > 0$ and the market price of risk λ are constant. The Sharpe ratio of the risky asset is therefore constant, and all time variation in investment opportunities comes through the interest rate, which follows the Gaussian dynamics of Vasicek (1977). When $r_1 > 0$, a high x means a high interest rate, that is, better investment opportunities.

The state and the asset are driven by the same Brownian motion, so there is a single source of risk, and because the asset loads on it the market is complete. The risky asset is only a vehicle for exposure to dB: any traded asset that spans the shock would do, with the portfolio weight rescaled by its volatility.

An investor who holds the fraction α of wealth in the risky asset and consumes at rate c has wealth dynamics

$$dW = \big[W\big(r(x) + \alpha\sigma_S\lambda\big) - c\big]\, dt + W\alpha\sigma_S\, dB. \tag{4}$$

The investor has CRRA utility

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}, \qquad \gamma > 0,\ \gamma \neq 1,$$

and solves

$$\max_{\{c_s, \alpha_s\}_{s \geq t}} \mathrm{E}_t\left[\int_t^\infty e^{-\delta(s-t)} \frac{c_s^{1-\gamma}}{1-\gamma}\, ds\right]$$

subject to (4).

The Optimal Consumption Path

Because the market is complete, the SDF is unique. From Discount Factors in Continuous Time, its drift is minus the short rate and its diffusion is minus the market price of risk:

$$\frac{d\Lambda}{\Lambda} = -r(x)\,dt - \lambda\,dB. \tag{5}$$

The previous notebook showed that the investor's discounted marginal utility, $e^{-\delta t}u'(c_t^*)$, is also an SDF. Uniqueness means the two can differ only by a constant factor. Write C_t for the optimal consumption path c_t^*. Then $e^{-\delta t}C_t^{-\gamma} \propto \Lambda_t$, so $\ln C_t = -(\ln \Lambda_t + \delta t)/\gamma$ up to a constant, and Ito's lemma gives

$$d\ln C = (\mu_0 + \mu_1 x)\,dt + \sigma_c\,dB, \tag{6}$$

with

$$\sigma_c = \frac{\lambda}{\gamma}, \qquad \mu_0 = \frac{r_0 - \delta}{\gamma} + \frac{\lambda^2}{2\gamma}, \qquad \mu_1 = \frac{r_1}{\gamma}. \tag{7}$$

The investor takes on consumption risk in proportion to its price λ, scaled down by risk aversion, and lets consumption grow faster when interest rates are high, again scaled by $1/\gamma$, the elasticity of intertemporal substitution. These are properties of this investor's choice: another investor with a different γ or δ, facing the same prices, chooses a different consumption path.

It is convenient to work with (μ_0, μ_1, σ_c) instead of (r_0, r_1, λ), since the moments of consumption growth are what the value function depends on. Inverting (7),

$$r(x) = \delta + \gamma(\mu_0 + \mu_1 x) - \frac{1}{2}\gamma^2\sigma_c^2, \qquad \lambda = \gamma\sigma_c, \tag{8}$$

which is the familiar consumption-based formula for the interest rate, here read in reverse: the prices determine the consumption path, not the other way around.

In a complete market the dynamic budget constraint (4), together with a transversality condition, is equivalent to a single static constraint (Cox and Huang 1989): wealth equals the value of future consumption. Along the optimal path,

$$W_t = \mathrm{E}_t\left[\int_0^\infty \frac{\Lambda_{t+\tau}}{\Lambda_t}\, C_{t+\tau}\, d\tau\right]. \tag{9}$$

Wealth-Consumption Ratio

Let

$$p(x) \equiv \frac{W}{C}$$

denote the wealth-consumption ratio along the optimal path. Substituting $\Lambda_{t+\tau}/\Lambda_t = e^{-\delta\tau}(C_{t+\tau}/C_t)^{-\gamma}$ into (9) and dividing by C_t gives

$$p(x_t) = \mathrm{E}_t\left[\int_0^\infty e^{-\delta\tau}\left(\frac{C_{t+\tau}}{C_t}\right)^{1-\gamma} d\tau\right]. \tag{10}$$

Because x is an Ornstein-Uhlenbeck process, the log consumption increment is conditionally normal. Define

$$\ell(\tau) \equiv \frac{1-e^{-\kappa\tau}}{\kappa}, \qquad a \equiv \frac{\mu_1\sigma_x}{\kappa}.$$

Integrating (6), using the solution $x_u = \bar{x} + (x_t - \bar{x})e^{-\kappa(u-t)} + \sigma_x \int_t^u e^{-\kappa(u-s)}\, dB_s$ of (1), gives

$$\Delta c_{t,\tau} \equiv \ln\left(\frac{C_{t+\tau}}{C_t}\right) = \mu_0\tau + \mu_1[\bar{x}\tau + (x_t - \bar{x})\ell(\tau)] + \int_t^{t+\tau}\left[\sigma_c + a\left(1 - e^{-\kappa(t+\tau-u)}\right)\right] dB_u.$$

A shock at time u moves log consumption directly through σ_c and indirectly through its effect on expected growth over the remaining horizon. For shocks far from the end of the horizon the coefficient approaches $\sigma_c + a$, so a captures the long-run contribution of the state variable to consumption volatility.

Hence $\Delta c_{t,\tau} \mid x_t$ is normal with mean

$$m_c(\tau, x_t) = \mu_0\tau + \mu_1[\bar{x}\tau + (x_t - \bar{x})\ell(\tau)],$$

and variance

$$v_c(\tau) = (\sigma_c + a)^2\tau - 2a(\sigma_c + a)\ell(\tau) + \frac{a^2}{2\kappa}\left(1 - e^{-2\kappa\tau}\right).$$

Using the lognormal moment formula in (10) gives the exact ratio

$$p(x) = \int_0^\infty \exp\!\big(\mathcal{A}(\tau) + \mathcal{B}(\tau)x\big)\, d\tau, \tag{11}$$

where

$$\mathcal{B}(\tau) = (1-\gamma)\mu_1\ell(\tau),$$

and

$$\mathcal{A}(\tau) = -\delta\tau + (1-\gamma)\big[\mu_0\tau + \mu_1\bar{x}\big(\tau - \ell(\tau)\big)\big] + \frac{1}{2}(1-\gamma)^2 v_c(\tau).$$

The integral converges when the long-run exponent is negative:

$$-\delta + (1-\gamma)(\mu_0 + \mu_1\bar{x}) + \frac{1}{2}(1-\gamma)^2\left(\sigma_c + \frac{\mu_1\sigma_x}{\kappa}\right)^2 < 0. \tag{12}$$

Economically, this condition ensures that the optimal consumption stream has finite value; if it fails, the expected utility integral diverges and no optimum exists. Since μ_0, μ_1 and σ_c depend on γ through (7), the condition restricts which investors have a well-defined problem in a given market.

Equation (11) is the continuous-time analogue of the wealth-consumption ratio in the discrete-time lognormal consumption model, with the sum over horizons replaced by an integral. The state enters only through the affine loading $\mathcal{B}(\tau)x$; the remaining term $\mathcal{A}(\tau)$ depends only on horizon τ. Wachter (2002) obtains a solution of the same form when the market price of risk, rather than the interest rate, follows an Ornstein-Uhlenbeck process.

Property 1 (Exact Wealth-Consumption Ratio). *In the one-factor Gaussian model,*

$$p(x) = \frac{W}{C} = \int_0^\infty \exp\!\big(\mathcal{A}(\tau) + \mathcal{B}(\tau)x\big)\, d\tau,$$

provided the finiteness condition (12) holds.

Value Function and Optimal Consumption

Along the optimal path, the value function is the expected utility of the optimal consumption stream, which we can compute directly with the same moment formula:

$$V(W_t, x_t) = \mathsf{E}_t\left[\int_0^\infty e^{-\delta\tau} \frac{C_{t+\tau}^{1-\gamma}}{1-\gamma}\, d\tau\right] = \frac{C_t^{1-\gamma}}{1-\gamma}\, p(x_t).$$

Substituting $C_t = W_t/p(x_t)$ gives

$$V(W, x) = \frac{W^{1-\gamma}}{1-\gamma} p(x)^{\gamma}. \tag{13}$$

This holds for every (W, x), since for a given x the level of optimal consumption, and hence of wealth, can be anything: it scales with the investor's initial wealth.

The consumption FOC from the previous notebook, $u'(c) = V_W(W, x)$, confirms this. Homotheticity of CRRA preferences suggests the guess $V(W, x) = W^{1-\gamma} h(x)/(1-\gamma)$, so that $V_W = W^{-\gamma} h(x)$. The FOC then reads $c^{-\gamma} = W^{-\gamma} h(x)$, and at the optimum $c = C = W/p(x)$ it requires $h(x) = p(x)^{\gamma}$, as in (13).

The FOC also delivers the optimal consumption rule:

$$c^*(W, x) = \frac{W}{p(x)}. \tag{14}$$

The investor consumes a state-dependent fraction $1/p(x)$ of wealth.

Optimal Portfolio

From the HJB first-order condition derived in the previous notebook, the Merton rule with one risky asset and one state variable perfectly correlated with it gives

$$\alpha^* = \frac{\lambda}{\gamma \sigma_S} + \frac{\sigma_x}{\gamma \sigma_S} \frac{\partial \ln V_W(W,x)}{\partial x}.$$

Since

$$V_W(W,x) = W^{-\gamma} p(x)^{\gamma},$$

we have

$$\frac{\partial \ln V_W(W,x)}{\partial x} = \gamma \frac{p'(x)}{p(x)},$$

and with $\lambda = \gamma \sigma_c$ the optimal portfolio simplifies to

$$\alpha^* = \frac{\sigma_c}{\sigma_S} + \frac{\sigma_x}{\sigma_S} \frac{p'(x)}{p(x)}. \tag{15}$$

Differentiating (11) under the integral sign yields

$$p'(x) = \int_0^\infty \mathcal{B}(\tau) \exp\big(\mathcal{A}(\tau) + \mathcal{B}(\tau)x\big)\, d\tau,$$

and therefore

$$\alpha^* = \frac{\sigma_c}{\sigma_S} + \frac{\sigma_x}{\sigma_S} \frac{\int_0^\infty \mathcal{B}(\tau) \exp\big(\mathcal{A}(\tau) + \mathcal{B}(\tau)x\big)\, d\tau}{\int_0^\infty \exp\big(\mathcal{A}(\tau) + \mathcal{B}(\tau)x\big)\, d\tau}. \tag{16}$$

The ratio p'/p is a weighted average of the loadings $\mathcal{B}(\tau)$, with weights proportional to the contribution of each horizon to wealth. It therefore lies between 0 and $\mathcal{B}(\infty) = (1-\gamma)\mu_1/\kappa$ and has the sign of $(1-\gamma)\mu_1$, which is the sign of $(1-\gamma)r_1$.

Property 2 (Portfolio Rule)**.** *The optimal risky share is*

$$\alpha^* = \underbrace{\frac{\lambda}{\gamma \sigma_S}}_{\textit{myopic demand}} + \underbrace{\frac{\sigma_x}{\sigma_S} \frac{p'(x)}{p(x)}}_{\textit{hedging demand}}.$$

The first term is the standard Merton myopic demand, which equals σ_c/σ_S. The second term is the hedging demand, and its sign is the sign of $(1-\gamma)r_1\sigma_x$. Take the usual case $\gamma > 1$ and $r_1, \sigma_x > 0$. The risky asset then pays off when x rises, which is when interest rates rise and investment opportunities improve. An investor with $\gamma > 1$ values wealth most when opportunities are poor, since $V_W = W^{-\gamma}p^{\gamma}$ is decreasing in x, so the asset is a bad hedge and the investor holds less of it than the myopic demand. With $\gamma < 1$ the sign flips and the investor holds more. Brennan and Xia (2000) study this interest-rate hedging demand for an investor who holds stocks and bonds when interest rates are Gaussian.

Wealth Dynamics

We now derive the dynamics of the investor's wealth along the optimal path directly from $W = Cp(x)$. This gives a more compact expression for α^* and a check that p values the consumption stream correctly.

Taking logs, $\ln W = \ln C + \ln p(x)$. Applying Ito's lemma with (6) and (1) implies

$$d\ln W = \left[\mu_0 + \mu_1 x + \kappa(\bar{x}-x)\frac{p'(x)}{p(x)} + \frac{1}{2}\sigma_x^2\left(\frac{p''(x)}{p(x)} - \left(\frac{p'(x)}{p(x)}\right)^2\right)\right]dt$$

$$+\left[\sigma_c + \sigma_x\frac{p'(x)}{p(x)}\right]dB,$$

and adding one half of the squared diffusion to the drift gives dW/W.

Property 3 (Wealth Dynamics)**.** *Along the optimal path, the investor's wealth satisfies*

$$\frac{dW}{W} = \mu_W(x)\,dt + \sigma_W(x)\,dB,$$

where

$$\mu_W(x) \equiv \mu_0 + \mu_1 x + \kappa(\bar{x}-x)\frac{p'(x)}{p(x)} + \frac{1}{2}\sigma_x^2\left(\frac{p''(x)}{p(x)} - \left(\frac{p'(x)}{p(x)}\right)^2\right) + \frac{1}{2}\left(\sigma_c + \sigma_x\frac{p'(x)}{p(x)}\right)^2,$$

and

$$\sigma_W(x) = \sigma_c + \sigma_x \frac{p'(x)}{p(x)}.$$

Comparing the diffusion term with (4), the portfolio rule can be written equivalently as

$$\alpha^* = \frac{\sigma_W(x)}{\sigma_S}.$$

The investor chooses exactly the risky exposure that gives wealth the volatility σ_W needed to finance the optimal consumption path, and holds the rest of wealth in the money-market account.

The hedging term can dominate. When $\sigma_x p'/p < -\sigma_c$, wealth has negative volatility: a positive shock raises consumption, but it also raises interest rates by enough that the present value of future consumption falls. The investor then shorts the risky asset, $\alpha^* < 0$.

Checking the Pricing Equation

Viewed as a claim to the investor's own consumption stream, wealth pays the dividend yield $q = C/W = 1/p(x)$. The fundamental pricing equation of Discount Factors in Continuous Time states that its expected total return in excess of the short rate equals minus its covariance with the SDF (5):

$$\mu_W(x) + \frac{1}{p(x)} - r(x) = \lambda \sigma_W(x). \tag{17}$$

This must hold if p is the value computed in (11). Substituting μ_W, σ_W and (8), the $(p'/p)^2$ terms cancel, and multiplying by p leaves the linear ODE

$$\frac{1}{2}\sigma_x^2 p''(x) + \left[\kappa(\bar{x} - x) + (1-\gamma)\sigma_c\sigma_x\right]p'(x) + \left[(1-\gamma)(\mu_0 + \mu_1 x) + \frac{1}{2}(1-\gamma)^2\sigma_c^2 - \delta\right]p(x) + 1 = 0. \tag{18}$$

This is the Feynman-Kac equation (Shreve 2004) for the expectation (10). Write $\mathcal{L}$ for the left-hand side of (18) without the constant 1. Each term $f(\tau, x) = \exp(\mathcal{A}(\tau) + \mathcal{B}(\tau)x)$ of the integral satisfies $\mathcal{L}f = \partial f/\partial\tau$, so integrating over τ gives $\mathcal{L}p = f(\infty, x) - f(0, x) = -1$, which is (18). The integral formula and the pricing equation therefore agree. The extra drift term

$(1-\gamma)\sigma_c\sigma_x$ reflects the covariance between the state and the weight $(C_{t+\tau}/C_t)^{1-\gamma}$ that the investor places on future consumption.

Example 1. Let the market have $r_0 = 0.055$, $r_1 = 5$, $\lambda = 0.1$, $\sigma_S = 0.2$, $\bar{x} = 0$, $\sigma_x = 0.005$, and $\kappa = 0.2$, and consider an investor with $\gamma = 5$ and $\delta = 0.01$. Evaluate everything at $x = \bar{x}$, where $r = 5.5\%$. By (7), the investor's optimal consumption has

$$\sigma_c = \frac{0.1}{5} = 0.02, \qquad \mu_0 = \frac{0.055 - 0.01}{5} + \frac{0.01}{10} = 0.01, \qquad \mu_1 = \frac{5}{5} = 1.$$

Then $a = 1 \times 0.005/0.2 = 0.025$, and the left-hand side of (12) is

$$-0.01 + (-4)(0.01) + \tfrac{1}{2}(16)(0.045)^2 = -0.0338 < 0,$$

so p is finite.

Evaluating (11) numerically gives $p(\bar{x}) \approx 27.7$, a consumption-wealth ratio of about 3.6%. The loadings $\mathcal{B}(\tau) = -4\ell(\tau)$ fall from 0 toward $\mathcal{B}(\infty) = -20$, and their weighted average is $p'/p \approx -17.0$. Hence

$$\sigma_W = 0.02 + 0.005 \times (-17.0) \approx 0.02 - 0.085 = -0.065.$$

The portfolio demands are

$$\frac{\lambda}{\gamma\sigma_S} = 0.10, \qquad \frac{\sigma_x}{\sigma_S}\frac{p'}{p} \approx -0.425, \qquad \alpha^* \approx -0.325.$$

The hedging demand is more than four times the myopic demand and has the opposite sign, so the investor shorts the risky asset. By (17), the investor's wealth earns a negative risk premium of $\lambda\sigma_W \approx -0.65\%$: it pays off when marginal utility is high.

Other investors in the same market choose differently. With $\gamma = 2$ the myopic demand is 0.25 and the hedging demand is about -0.27, so $\alpha^* \approx -0.02$. With $\gamma = 10$ the myopic demand is 0.05 and the hedging demand is about -0.48, so $\alpha^* \approx -0.43$. An investor with $\gamma = 0.5$ violates (12) in this market and has no optimum. □

As risk aversion rises, the myopic demand shrinks toward zero while the hedging demand grows in magnitude. A very risk-averse investor mostly uses the risky asset to offset interest-rate risk, that is, to lock in a smooth consumption stream, much as a long-term bond would. Campbell and Viceira (2001) and Brennan and Xia (2002) make this point for long-term inflation-indexed bonds, which are the riskless asset for a long-horizon investor.

The negative premium does not mean consumption risk is unpriced. A positive shock raises consumption, which lowers marginal utility, but it also raises x and hence the discount rate applied to every future payout. With a persistent state (κ small) and $\gamma > 1$, the discount-rate effect can outweigh the cash-flow effect, as it does in the example.

Special Cases

When $r_1 = 0$, the interest rate is constant and so is optimal expected consumption growth, $\mu_1 = 0$. Then $\mathcal{B}(\tau) = 0$, so $p(x)$ is constant, $p'(x) = 0$, and the state variable drops out of the solution. The model collapses to the standard Merton benchmark:

$$\frac{W}{C} = \text{constant}, \qquad \alpha^* = \frac{\lambda}{\gamma \sigma_S}.$$

There is no hedging demand because investment opportunities no longer vary over time.

The case $\gamma = 1$ requires separate treatment because the CRRA formula is undefined there; log utility $u(c) = \ln c$ is its limit, and the value function takes the form $V(W, x) = \ln W/\delta + h(x)$. The wealth-consumption ratio, however, follows by taking the limit in the integrand: $\mathcal{B}(\tau) = 0$ and $\mathcal{A}(\tau) = -\delta\tau$, so

$$p(x) = \int_0^\infty e^{-\delta\tau}\, d\tau = \frac{1}{\delta}.$$

The wealth-consumption ratio is constant and equal to $1/\delta$, independent of the state variable, so there is again no hedging demand.

Conclusion

This one-factor Gaussian specification is the simplest exact state-dependent solution to the continuous-time consumption-portfolio problem with additive utility. The main objects all reduce to the wealth-consumption ratio $p(x)$:

$$V(W,x) = \frac{W^{1-\gamma}}{1-\gamma}p(x)^{\gamma}, \qquad c^* = \frac{W}{p(x)}, \qquad \alpha^* = \frac{\lambda}{\gamma\sigma_S} + \frac{\sigma_x}{\sigma_S}\frac{p'(x)}{p(x)}.$$

The state variable matters only through how it changes the present value of future consumption. That is exactly the hedging channel in Merton's ICAPM: when the state changes the entire future path of investment opportunities, optimal portfolio choice depends not only on today's Sharpe ratio, but also on how wealth responds to that state.

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