

FX Option Pricing Models

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Continuous-Time Model of Exchange Rates

Let S_t denote the exchange rate between two currencies, and let r_t and r_t^F be the risk-free interest rates in the domestic and foreign country, respectively. Throughout, T denotes a fixed expiry date shared by all bonds and the option, so we suppress the argument (T) from bond prices, forward prices, and related quantities. We define

- Z_t : the price (in the domestic currency) of a zero-coupon bond paying 1 unit of the domestic currency at time T ,
- Z_t^F : the price (in the foreign currency) of a zero-coupon bond paying 1 unit of the foreign currency at time T .

The forward exchange rate is then

$$F_t = \frac{S_t Z_t^F}{Z_t}.$$

Consider a call option on S_t with strike K expiring at time T . Let C_t be the price of the option at time t , and define the money-market account $\beta_t = e^{\int_0^t r_s ds}$. At expiration,

$$C_T = (S_T - K)^+.$$

Under the risk-neutral measure P^* , defined by the SDF Λ_t via $\mathcal{E}_t = \Lambda_t \beta_t / (\Lambda_0 \beta_0)$,

$$\frac{C_t}{\beta_t} = E_t^* \left(\frac{C_T}{\beta_T} \right).$$

The T-Forward Measure

The general principle, following Geman et al. (1995), is that any strictly positive asset price N_t defines a measure P_N under which all N -discounted prices are martingales. Changing from a measure with numeraire N_t^A to one with numeraire N_t^B corresponds to the Radon-Nikodym derivative

$$\frac{d P_{N^B}}{d P_{N^A}} = \frac{N_T^B / N_0^B}{N_T^A / N_0^A},$$

i.e., the ratio of the two numeraires, each normalized by its initial value. For example, setting $N^A = \beta$ and $N^B = \Lambda\beta$ (the SDF times the money-market account, unnormalized) recovers the risk-neutral measure from the physical measure. Setting $N^B = \Lambda S$ gives the stock measure P^S of the continuous-time pricing notebook, and so on.

We introduce the domestic T-forward measure P_T by choosing $N^A = \beta_t$ and $N^B = Z_t$, giving the strictly positive P^* -martingale

$$\mathcal{E}_t^Z = \frac{Z_t / \beta_t}{Z_0 / \beta_0},$$

and setting $d P_T / d P^* = \mathcal{E}_T^Z$. The intuition is that Z_t / β_t is the discounted bond price, a P^* -martingale. This changes the numeraire from β_t to Z_t : the risk-neutral pricing equation $V_t / \beta_t = E_t^*(V_T / \beta_T)$ becomes $V_t / Z_t = E_t^T(V_T / Z_T)$ under P_T .

Under P_T , the forward price F_t is a martingale. Indeed,

$$F_t = \frac{S_t Z_t^F}{Z_t} = E_t^T \left(\frac{S_T Z_T^F}{Z_T} \right) = E_t^T(S_T),$$

so the forward rate equals the expected spot rate under P_T .

Using this change of measure, the call option price satisfies

$$\begin{aligned} \frac{C_t}{\beta_t} &= E_t^* \left(\frac{C_T}{\beta_T} \right) \\ &= E_t^T \left(\frac{d P^*}{d P_T} \frac{C_T}{\beta_T} \right) \\ &= E_t^T \left(\frac{\beta_T / \beta_t}{Z_T / Z_t} \frac{C_T}{\beta_T} \right), \end{aligned}$$

and therefore

$$\begin{aligned}\frac{C_t}{Z_t} &= \mathbb{E}_t^T \left(\frac{C_T}{Z_T} \right) \\ &= \mathbb{E}_t^T ((S_T - K)^+) \\ &= \mathbb{E}_t^T (S_T \mathbf{1}_{\{S_T > K\}}) - K \mathbb{E}_t^T (\mathbf{1}_{\{S_T > K\}}).\end{aligned}$$

To evaluate the first expectation, let X_t be the time- t price of the security that pays $X_T = S_T \mathbf{1}_{\{S_T > K\}}$ at time T . Define the foreign T -forward measure $\mathbb{P}_T^F$ via the strictly positive $\mathbb{P}_T$ -martingale

$$\mathcal{E}_t^F = \frac{F_t}{F_0},$$

setting $d\mathbb{P}_T^F / d\mathbb{P}_T = \mathcal{E}_T^F$. Since $F_T = S_T Z_T^F / Z_T$, we have $\mathcal{E}_T^F = (S_T Z_T^F / Z_T) / (S_0 Z_0^F / Z_0)$, which matches the Radon-Nikodym derivative written as

$$\frac{d\mathbb{P}_T^F}{d\mathbb{P}_T} = \frac{S_T Z_T^F / (S_0 Z_0^F)}{Z_T / Z_0}.$$

This is a further change of numeraire from Z_t to $S_t Z_t^F$, the domestic value of the foreign zero-coupon bond. Under $\mathbb{P}_T^F$, the inverse forward price $G_t = 1/F_t = Z_t / (S_t Z_t^F)$ is a martingale:

$$G_t = \mathbb{E}_t^F \left(\frac{Z_T}{S_T Z_T^F} \right) = \mathbb{E}_t^F \left(\frac{1}{S_T} \right).$$

This shows that $\mathbb{P}_T$ is the natural measure for domestic investors while $\mathbb{P}_T^F$ is the natural measure for foreign investors.

Since X_t can be valued under either numeraire,

$$\frac{X_t}{Z_t} = \mathbb{E}_t^T (S_T \mathbf{1}_{\{S_T > K\}}),$$

and

$$\frac{X_t}{S_t Z_t^F} = \mathbb{E}_t^F (\mathbf{1}_{\{S_T > K\}}) = \mathbb{P}_T^F(S_T > K).$$

Combining these results gives the general FX option pricing formula.

Property 1. *The time- t price of a European call on the exchange rate with strike K and expiry T is*

$$C_t = S_t Z_t^F P_T^F(S_T > K) - K Z_t P_T(S_T > K).$$

This result extends the formula of Geman et al. (1995) to the case of a dividend-paying asset, with the foreign interest rate playing the role of the dividend yield.

Deterministic Forward Volatility

We now assume that the forward price process satisfies

$$\frac{dF_t}{F_t} = \sigma_F(t) dB_t^T,$$

where B_t^T is a Brownian motion under P_T and $\sigma_F(t)$ is a deterministic function of time. Because instantaneous volatility is unchanged under equivalent measure changes, the inverse forward price satisfies

$$\frac{dG_t}{G_t} = \sigma_F(t) d\hat{B}_t^T,$$

where $\hat{B}_t^T$ is a Brownian motion under P_T^F .

Since F_t is a P_T -martingale with log-normal dynamics,

$$\ln F_T = \ln F_0 - \frac{1}{2} \int_0^T \sigma_F^2(s) ds + \int_0^T \sigma_F(s) dB_s^T.$$

Standardizing the log-normal distribution in the usual way,

$$\begin{aligned} P_T(S_T > K) &= P_T(F_T > K) \\ &= N\left(\frac{\ln(F_0/K) - \frac{1}{2} \int_0^T \sigma_F^2(s) ds}{\sqrt{\int_0^T \sigma_F^2(s) ds}}\right). \end{aligned}$$

Similarly, since G_t is a P_T^F -martingale,

$$\begin{aligned} P_T^F(S_T > K) &= P_T^F(F_T > K) \\ &= N\left(\frac{\ln(F_0/K) + \frac{1}{2} \int_0^T \sigma_F^2(s) ds}{\sqrt{\int_0^T \sigma_F^2(s) ds}}\right). \end{aligned}$$

Substituting into Property 1 yields the following result.

Property 2. *Under deterministic forward volatility, the call price is*

$$C_0 = S_0 Z_0^F N(d_1) - K Z_0 N(d_2),$$

where

$$d_1 = \frac{\ln(F_0/K) + \frac{1}{2} \int_0^T \sigma_F^2(s) ds}{\sqrt{\int_0^T \sigma_F^2(s) ds}}, \quad d_2 = d_1 - \sqrt{\int_0^T \sigma_F^2(s) ds}.$$

Constant Interest Rates and Volatility

We now recover the Garman and Kohlhagen (1983) FX option valuation formula. Assume that the spot rate follows a geometric Brownian motion under the risk-neutral measure,

$$\frac{dS_t}{S_t} = (r - r^F) dt + \sigma_S dB_t^*,$$

where r and r^F are constant domestic and foreign interest rates and σ_S is the constant volatility of spot-rate returns. In this case,

$$F_t = S_t e^{(r-r^F)(T-t)}, \quad Z_t^F = e^{-r^F(T-t)}, \quad Z_t = e^{-r(T-t)}.$$

The dynamics of the forward rate equal those of the spot rate, so $\sigma_F^2(t) = \sigma_S^2$ and $\int_0^T \sigma_F^2(s) ds = \sigma_S^2 T$.

Property 3. In the Garman-Kohlhagen model, the price of a European call on the exchange rate is

$$C_0 = S_0 e^{-r^F T} N(d_1) - K e^{-r^T} N(d_2),$$

where

$$d_1 = \frac{\ln(S_0/K) + (r - r^F + \frac{1}{2}\sigma_S^2) T}{\sigma_S \sqrt{T}}, \quad d_2 = d_1 - \sigma_S \sqrt{T}.$$

Stochastic Interest Rates and Constant Volatility

We now consider an example with stochastic interest rates in both countries, analogous to the model of Schwartz (1997) for commodities. The spot rate follows a geometric Brownian motion and the two interest rates follow Ornstein-Uhlenbeck processes:

$$\begin{aligned} \frac{dS_t}{S_t} &= (r_t - r_t^F) dt + \sigma_1 dB_{1,t}, \\ dr_t &= a(m - r_t) dt + \sigma_2 dB_{2,t}, \\ dr_t^F &= k(\alpha - r_t^F) dt + \sigma_3 dB_{3,t}, \end{aligned}$$

where $dB_{1,t} dB_{2,t} = \rho_1 dt$, $dB_{2,t} dB_{3,t} = \rho_2 dt$, and $dB_{1,t} dB_{3,t} = \rho_3 dt$.

Under this model, the domestic and foreign bond prices admit the closed-form expressions

$$\begin{aligned} Z_0 = \exp & \left(-r_0 \frac{1 - e^{-aT}}{a} \right. \\ & + \frac{m((1 - e^{-aT}) - aT)}{a} \\ & \left. - \frac{\sigma_2^2 (4(1 - e^{-aT}) - (1 - e^{-2aT}) - 2aT)}{4a^3} \right), \end{aligned}$$

and

$$Z_0^F = \exp\left(-r_0^F \frac{1 - e^{-kT}}{k} + \frac{(k\alpha + \sigma_1\sigma_3\rho_3)((1 - e^{-kT}) - kT)}{k^2} - \frac{\sigma_3^2(4(1 - e^{-kT}) - (1 - e^{-2kT}) - 2kT)}{4k^3}\right).$$

The instantaneous variance of the forward rate under P_T is

$$\sigma_F^2(t) = \sigma_1^2 + f(t)^2\sigma_2^2 + 2f(t)\sigma_1\sigma_2\rho_1 + g(t)^2\sigma_3^2 - 2g(t)\sigma_1\sigma_3\rho_3 - 2f(t)g(t)\sigma_2\sigma_3\rho_2,$$

where

$$f(t) = \frac{1 - e^{-a(T-t)}}{a}, \quad g(t) = \frac{1 - e^{-k(T-t)}}{k}.$$

The call price then follows from Property 2 by substituting $\int_0^T \sigma_F^2(s) ds$, obtained by integrating the expression above from 0 to T :

$$C_0 = S_0 Z_0^F N(d_1) - K Z_0 N(d_2),$$

where d_1 and d_2 are as in Property 2 with the integrated forward variance in place of $\sigma_S^2 T$.

References

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