

Recursive Preferences in a Multiperiod Economy

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Introduction

The [Epstein-Zin preferences](#) notebook derived the SDF in a two-date setting by guessing a homothetic consumption rule and solving a single maximization problem. That approach is clean but sidesteps the recursive structure that makes Epstein-Zin preferences genuinely useful in multiperiod environments. This notebook derives the same SDF directly from the multiperiod recursive utility aggregator, relying on three tools: implicit differentiation of the recursive equation, a perturbation argument to extract the pricing kernel, and Euler's homogeneous function theorem to translate utility into observable quantities.

The payoff is the standard Epstein-Zin stochastic discount factor

$$m_{t+1} = \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-1/\psi} \right]^\theta \left[\frac{1}{R_{t+1}^w} \right]^{1-\theta}, \quad \theta = \frac{1-\gamma}{1-1/\psi},$$

expressed entirely in terms of consumption growth and the return on the wealth portfolio. The parameter θ controls the weight of each factor: when $\psi = 1/\gamma$, $\theta = 1$ and the formula collapses to the standard CRRA stochastic discount factor; otherwise the wealth return enters as a separate pricing factor, capturing the agent's concern for future investment opportunities — the source of Epstein-Zin's intertemporal hedging demand. The multiperiod derivation confirms that the two-date formula is exact in every period and not merely an approximation valid near a steady state.

General Recursive Utility

Kreps and Porteus (1978) develop a framework for recursive utility that separates attitudes toward risk from the willingness to substitute consumption over time. A general Kreps–Porteus aggregator takes the form

$$V_t = U(c_t, \mu_t),$$

where $U(c, \mu)$ is an increasing aggregator and μ_t is the certainty equivalent of next-period utility under a von Neumann–Morgenstern risk aggregator:

$$\mu_t = \phi^{-1}(E_t[\phi(V_{t+1})]),$$

for some increasing, concave ϕ . The concavity of ϕ makes the agent averse to uncertainty about future utility levels, while the curvature of U in its first argument separately controls the willingness to substitute consumption across time. This is the key departure from standard expected utility, where a single function governs both simultaneously. Epstein–Zin preferences (Epstein and Zin 1989) are the leading special case, with $U(c, \mu) = [(1 - \beta)c^\rho + \beta\mu^\rho]^{1/\rho}$ and $\phi(v) = v^{1-\gamma}$.

To derive the SDF, consider a small perturbation in the consumption plan: give up ξp_t units of consumption today in exchange for ξx_{t+1} units tomorrow. At the optimum the agent is indifferent at the margin, so the derivative of V_t with respect to ξ vanishes at $\xi = 0$. Differentiating $U(c_t - \xi p_t, \mu_t(\xi))$ at $\xi = 0$ and using $\partial V_{t+1}/\partial c_{t+1} = U_c(c_{t+1}, \mu_{t+1})$ recursively gives the first-order condition

$$p_t U_c(c_t, \mu_t) = U_\mu(c_t, \mu_t) \frac{E_t[\phi'(V_{t+1}) U_c(c_{t+1}, \mu_{t+1}) x_{t+1}]}{\phi'(\mu_t)},$$

so the SDF is

$$m_{t+1} = \underbrace{\frac{U_\mu(c_t, \mu_t)}{U_c(c_t, \mu_t)}}_{\text{aggregator ratio}} \cdot \underbrace{\frac{\phi'(V_{t+1})}{\phi'(\mu_t)}}_{\text{risk adjustment}} \cdot \underbrace{U_c(c_{t+1}, \mu_{t+1})}_{\text{marginal utility}}. \quad (1)$$

The middle factor $\phi'(V_{t+1})/\phi'(\mu_t)$ compares realized utility to the certainty equivalent through the lens of ϕ . For power $\phi(v) = v^{1-\gamma}$ this is $(V_{t+1}/\mu_t)^{-\gamma}$; for entropic $\phi(v) = -e^{-\alpha v}$ it becomes $e^{-\alpha(V_{t+1}-\mu_t)}$.

Epstein-Zin Preferences

The agent has preferences over consumption streams $(c_t, c_{t+1}, c_{t+2}, \dots)$ of the Kreps–Porteus form $V_t = U(c_t, \mu_t)$, with aggregator $U(c, \mu) = [(1-\beta)c^\rho + \beta\mu^\rho]^{1/\rho}$ and power risk aggregator $\phi(v) = v^{1-\gamma}$, so the certainty equivalent is

$$\mu_t = \left[E_t(V_{t+1}^{1-\gamma}) \right]^{1/(1-\gamma)}.$$

The parameters $\beta \in (0, 1)$, $\gamma > 0$, and $\psi > 0$ are the time-discount factor, coefficient of relative risk aversion, and elasticity of intertemporal substitution. Defining $\rho = 1 - 1/\psi$, the aggregator takes the explicit form

$$V_t = [(1 - \beta) c_t^\rho + \beta \mu_t^\rho]^{1/\rho}, \quad (2)$$

or equivalently,

$$V_t^\rho = (1 - \beta) c_t^\rho + \beta \mu_t^\rho. \quad (3)$$

When $\psi = 1/\gamma$ the aggregator collapses to standard CRRA utility with coefficient γ .

Homotheticity and the Wealth Identity

The aggregator (2) is linearly homogeneous in consumption: scaling every current and future consumption level by $\lambda > 0$ scales V_t by λ . Euler's theorem therefore gives

$$V_t = \sum_{\tau=0}^{\infty} \frac{\partial V_t}{\partial c_{t+\tau}} c_{t+\tau} = E_t \left[\sum_{\tau=0}^{\infty} \frac{\partial V_t}{\partial c_{t+\tau}} c_{t+\tau} \right],$$

where the second equality holds because the sum equals V_t state by state, hence in conditional expectation as well. Dividing by $U_c(c_t, \mu_t)$, the right-hand side is the present value of the entire consumption stream — total wealth W_t :

$$\frac{V_t}{U_c(c_t, \mu_t)} = E_t \left[\sum_{\tau=0}^{\infty} \frac{U_c(c_{t+\tau}, \mu_{t+\tau})}{U_c(c_t, \mu_t)} c_{t+\tau} \right] = W_t. \quad (4)$$

Substituting $U_c(c_t, \mu_t) = (1 - \beta) c_t^{-1/\psi} V_t^{1/\psi}$ into $V_t = U_c(c_t, \mu_t) W_t$ and simplifying:

$$V_t = (1 - \beta) c_t^{-1/\psi} V_t^{1/\psi} W_t \Rightarrow V_t^\rho = (1 - \beta) c_t^{-1/\psi} W_t, \quad (5)$$

so that

$$V_t = \left[(1 - \beta) c_t^{-1/\psi} W_t \right]^{1/\rho}. \quad (6)$$

The entire infinite-dimensional consumption stream is thus compressed into two quantities: current consumption c_t and total wealth W_t .

The Certainty Equivalent

Substituting (5) into the recursive equation (3) gives

$$(1 - \beta) c_t^{-1/\psi} W_t = (1 - \beta) c_t^\rho + \beta \mu_t^\rho.$$

Writing $(1 - \beta) c_t^\rho = (1 - \beta) c_t^{-1/\psi} \cdot c_t$ and rearranging:

$$(1 - \beta) c_t^{-1/\psi} (W_t - c_t) = \beta \mu_t^\rho.$$

Here $W_t - c_t$ is the amount saved after current consumption. Those savings are invested and grow to next-period wealth: $W_{t+1} = R_{t+1}^w (W_t - c_t)$, where R_{t+1}^w is the gross return on the wealth portfolio. Solving for the certainty equivalent:

$$\mu_t = \left[\frac{(1 - \beta) c_t^{-1/\psi} (W_t - c_t)}{\beta} \right]^{1/\rho}. \quad (7)$$

Deriving the Pricing Kernel

We apply (1) to the Epstein-Zin aggregator $U(c, \mu) = [(1 - \beta) c^\rho + \beta \mu^\rho]^{1/\rho}$ with risk aggregator $\phi(v) = v^{1-\gamma}$, for which $\mu_t = [E_t(V_{t+1}^{1-\gamma})]^{1/(1-\gamma)}$. The partial derivatives of U are

$$U_c = (1 - \beta) c^{\rho-1} U^{1-\rho}, \quad U_\mu = \beta \mu^{\rho-1} U^{1-\rho},$$

so their ratio is

$$\frac{U_\mu}{U_c} = \frac{\beta \mu^{\rho-1}}{(1-\beta) c^{\rho-1}}.$$

Since $\phi'(v) = (1-\gamma)v^{-\gamma}$, the risk-adjustment factor reduces to

$$\frac{\phi'(V_{t+1})}{\phi'(\mu_t)} = \left(\frac{V_{t+1}}{\mu_t} \right)^{-\gamma}.$$

Substituting into (1) and collecting terms using $\rho - 1 = -1/\psi$:

$$m_{t+1} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-1/\psi} \mu_t^{\rho-1+\gamma} V_{t+1}^{1/\psi-\gamma} = \beta \left(\frac{c_{t+1}}{c_t} \right)^{-1/\psi} \left(\frac{V_{t+1}}{\mu_t} \right)^{1/\psi-\gamma}. \quad (8)$$

This is an exact expression for the SDF in terms of the utility process; (6) and (7) now translate the ratio V_{t+1}/μ_t into observables:

$$\frac{V_{t+1}}{\mu_t} = \frac{\left[(1-\beta) c_{t+1}^{-1/\psi} W_{t+1} \right]^{1/\rho}}{\left[\frac{(1-\beta) c_t^{-1/\psi} (W_t - c_t)}{\beta} \right]^{1/\rho}} = \left[\beta \frac{W_{t+1}}{W_t - c_t} \left(\frac{c_{t+1}}{c_t} \right)^{-1/\psi} \right]^{1/\rho} = \left[\beta R_{t+1}^w \left(\frac{c_{t+1}}{c_t} \right)^{-1/\psi} \right]^{1/\rho},$$

where the last step uses $W_{t+1}/(W_t - c_t) = R_{t+1}^w$. Substituting into (8) and collecting terms:

$$m_{t+1} = \beta^{(1-\gamma)/\rho} (R_{t+1}^w)^{(1/\psi-\gamma)/\rho} \left(\frac{c_{t+1}}{c_t} \right)^{-(1-\gamma)/(\rho\psi)}.$$

Defining $\theta = (1-\gamma)/\rho = (1-\gamma)/(1-1/\psi)$, so that $1-\theta = (\gamma-1/\psi)/\rho$, the Epstein-Zin stochastic discount factor is

$$m_{t+1} = \beta^\theta \left(\frac{c_{t+1}}{c_t} \right)^{-\theta/\psi} (R_{t+1}^w)^{\theta-1}, \quad (9)$$

or equivalently,

$$m_{t+1} = \left[\beta \left(\frac{c_{t+1}}{c_t} \right)^{-1/\psi} \right]^\theta \left[\frac{1}{R_{t+1}^w} \right]^{1-\theta}, \quad \theta = \frac{1-\gamma}{1-1/\psi}. \quad (10)$$

The parameter θ measures the weight placed on the consumption-growth factor relative to the wealth-return factor. When $\theta = 1$ — which occurs exactly when $\psi = 1/\gamma$, the CRRA case — the wealth return drops out entirely. When $\theta \neq 1$, the wealth return enters the SDF as a separate pricing factor, reflecting the agent’s concern for news about future investment opportunities beyond what current consumption growth reveals. The further θ departs from one, the more the wealth return matters as an independent pricing factor.

References

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