Consumption Based Asset Pricing

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Introduction

The consumption-based approach to asset pricing derives equilibrium restrictions on asset prices from the optimizing behavior of a representative investor. The framework was developed by Lucas (1978) and Breeden (1979) and is the standard foundation for studying risk premia and equilibrium returns (Cochrane 2009).

In Consumption and Portfolio Choice, we derived the stochastic discount factor by solving a dynamic programming problem. The investor chose current consumption c_0 and portfolio weights $\boldsymbol{\alpha}_0$, and stacking the two first-order conditions — together with the envelope theorem — yielded

$$m = \beta \frac{u'(c_1)}{u'(c_0)}.$$

That derivation gave us the SDF as a byproduct of the full optimization. It also required setting up the Bellman equation and differentiating the value function.

This notebook derives the same result from a more direct angle: the perturbation argument. Rather than solving the full optimization problem, we take an already-optimal consumption plan as given and ask how utility changes if the investor deviates from it by purchasing a small amount of one additional asset. The first-order effect of this deviation on utility must be zero at the optimum, and the resulting condition is the Euler equation. The SDF emerges immediately as the ratio of marginal utilities. The argument is powerful precisely because it requires very little structure — we need only that the investor is optimizing over consumption, without specifying how savings are invested or how many periods remain.[1]

[1] This argument is also the discrete-time analogue of the calculus of variations. In the classical variational setting, one perturbs a candidate optimal path and sets the derivative of the resulting functional to zero; the necessary conditions are the Euler-Lagrange equations. The asset purchase plays the role of the path perturbation, and the connection is exact: the continuous-time version of this problem recovers the Euler-Lagrange equations directly (Lanczos 1970).

Once the SDF is in hand, we derive its asset pricing implications: how it prices any payoff, what it implies for the risk-free rate, and how expected returns relate to covariance with marginal utility.

The Perturbation Argument

Consider an investor with a two-date utility function

$$U(c_0, c_1) = u(c_0) + \beta\, \mathsf{E}[u(c_1)], \tag{1}$$

where $u(\cdot)$ is an increasing and concave function of consumption and $\beta < 1$ is a discount factor. Suppose the investor is already following an optimal consumption-investment plan that yields consumption (c_0, c_1).

We now ask: what if the investor deviates from this plan by purchasing ξ additional shares of an asset that trades at price p and delivers a (possibly random) payoff x? To finance the purchase, she reduces today's consumption by ξp. Tomorrow, she receives the additional payoff ξx. The utility of this deviation, as a function of ξ, is

$$V(\xi) = u(c_0 - \xi p) + \beta\, \mathsf{E}[u(c_1 + \xi x)].$$

Since (c_0, c_1) is optimal, $\xi = 0$ maximizes $V(\xi)$. The first-order condition at $\xi = 0$ is

$$-pu'(c_0) + \beta\, \mathsf{E}[xu'(c_1)] = 0,$$

which rearranges to

$$p = \mathsf{E}\left[\beta \frac{u'(c_1)}{u'(c_0)}\, x\right]. \tag{2}$$

Equation (2) is the Euler equation for asset prices. It says that any asset the investor can trade must be priced by the intertemporal marginal rate of substitution — the ratio $\beta u'(c_1)/u'(c_0)$, which measures how the investor values a unit of future consumption relative to a unit today.

Property 1 (Consumption-Based Pricing Kernel). *The Euler equation identifies the SDF as*

$$m = \beta \frac{u'(c_1)}{u'(c_0)}. \tag{3}$$

The price of any payoff x satisfies

$$p = \mathrm{E}(mx).$$

This is the same SDF derived in Consumption and Portfolio Choice from the dynamic programming approach — the two routes give identical objects. The perturbation argument is simply a more direct path to the Euler equation that does not require the Bellman machinery.

The economic interpretation is immediate. An asset that pays well when $u'(c_1)$ is high — that is, when future consumption is low and marginal utility is elevated — receives a large discount factor. Such an asset is valuable insurance, so investors bid up its price. Conversely, an asset that pays mainly in good times, when $u'(c_1)$ is low, provides little insurance and commands a lower price.

As in Stochastic Discount Factor, if x is defined on a finite probability space $(\mathcal{S}, q)$, the pricing equation becomes

$$p = \sum_{s \in \mathcal{S}} q(s)\, m(s)\, x(s). \tag{4}$$

Power Utility

Specializing to power utility $u(c) = c^{1-\gamma}/(1-\gamma)$, so that $u'(c) = c^{-\gamma}$, the SDF takes the form

$$m = \beta \left(\frac{c_1}{c_0}\right)^{-\gamma}.$$

The SDF is driven entirely by consumption growth. States in which consumption falls sharply receive high discount factors, making assets that pay off in downturns especially valuable.

Example 1. Consider an investor with current consumption $c_0 = 6.5$, risk aversion $\gamma = 4$, and discount factor $\beta = 0.95$. There are three scenarios — Boom, Normal, and Recession —

with probabilities, next-period consumption, and payoffs of two assets X and Y given by

$$q = \begin{bmatrix} 0.3 \\ 0.5 \\ 0.2 \end{bmatrix}, \quad c_1 = \begin{bmatrix} 9.0 \\ 6.7 \\ 5.4 \end{bmatrix}, \quad x = \begin{bmatrix} 9.8 \\ 8.3 \\ 6.5 \end{bmatrix}, \quad y = \begin{bmatrix} 6.0 \\ 5.0 \\ 7.1 \end{bmatrix}.$$

The SDF in each scenario is $m = 0.95\,(c_1/6.5)^{-4}$:

$$m = \begin{bmatrix} 0.258 \\ 0.842 \\ 1.994 \end{bmatrix}.$$

Using equation (4),

$$p(X) = 0.3(0.258)(9.8) + 0.5(0.842)(8.3) + 0.2(1.994)(6.5) = 6.85,$$

$$p(Y) = 0.3(0.258)(6.0) + 0.5(0.842)(5.0) + 0.2(1.994)(7.1) = 5.40.$$

Since $\mathrm{E}(X) = 8.39$ and $\mathrm{E}(Y) = 5.72$, the expected returns are $\mathrm{E}(r^X) = 22.5\%$ and $\mathrm{E}(r^Y) = 5.91\%$. □

Asset Pricing Implications

In the Stochastic Discount Factor notebook we showed that for any traded payoff x with price p, the gross return $R = x/p$ satisfies $\mathrm{E}(mR) = 1$, that the risk-free rate is $R^f = 1/\mathrm{E}(m)$, and that the risk premium on any asset satisfies

$$\mathrm{E}(R^i) - R^f = -\frac{\mathrm{Cov}(m, R^i)}{\mathrm{E}(m)}.$$

Those results hold for any valid SDF. Now that $m = \beta u'(c_1)/u'(c_0)$ has an explicit economic form, each of them acquires a concrete interpretation in terms of consumption.

Example 2. Using the data of Example 1, $\mathrm{E}(m) = 0.3(0.258) + 0.5(0.842) + 0.2(1.994) =$

0.8972, so

$$R^f = \frac{1}{0.8972} = 1.1146,$$

or a net return of 11.46% per period. □

The risk premium formula says that assets paying off when marginal utility is high — when future consumption is low — are valuable insurance and carry lower expected returns. The same covariance decomposition applied to prices rather than returns gives

$$\begin{aligned} p &= \mathsf{E}(mx) = \mathsf{E}(m)\ \mathsf{E}(x) + \mathsf{Cov}(m, x) \\ &= \frac{\mathsf{E}(x)}{R^f} + \mathsf{Cov}(m, x), \end{aligned} \tag{5}$$

separating the time value of money from the risk adjustment.

Rewriting the risk premium in beta form,

$$\mathsf{E}(R^i) - R^f = \beta_{i,m}\, \lambda_m, \tag{6}$$

where $\beta_{i,m} = \mathsf{Cov}(R^i, m) / \mathsf{V}(m)$ and $\lambda_m = -\mathsf{V}(m) / \mathsf{E}(m) < 0$. This is a beta pricing model in which the single factor is marginal utility growth. Assets that covary positively with the SDF have positive $\beta_{i,m}$, but since $\lambda_m < 0$ they carry a negative risk premium and earn expected returns below the risk-free rate. They are insurance assets: investors accept low returns in exchange for payoffs concentrated in bad times.

References

Breeden, Douglas T. 1979. "An Intertemporal Asset Pricing Model with Stochastic Consumption and Investment Opportunities." *Journal of Financial Economics* 7 (3): 265–96.

Cochrane, John. 2009. *Asset Pricing: Revised Edition*. Princeton university press.

Lanczos, Cornelius. 1970. *The Variational Principles of Mechanics*. 4th ed. University of Toronto Press.

Lucas, Robert E. 1978. “Asset Prices in an Exchange Economy.” *Econometrica* 46 (6): 1429–45.