Discount Factors in Continuous Time

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Price Processes

In the following, we work in a probability space $(\Omega, \mathcal{F}, P)$. Uncertainty is driven by K independent Brownian motions such that

$$(d\mathbf{B})(d\mathbf{B})' = \mathbf{I}dt,$$

where $\mathbf{B} = \begin{pmatrix} B_1 & B_2 & ... & B_K \end{pmatrix}'$ and $\mathbf{I}$ is a $K \times K$ identity matrix.

We can always generate correlated Brownian motions by combining two independent Brownian motions. For example, let $Z = \rho B_1 + \sqrt{1 - \rho^2} B_2$ where $|\rho| \leq 1$. Then, Z is a Brownian motion since $(dZ)^2 = \rho^2 dt + (1 - \rho^2)dt = dt$. Furthermore, Z and B_1 are instantaneously correlated since $(dZ)(dB_1) = \rho(dB_1)^2 = \rho dt$.

More generally, any Brownian motion Z correlated with $B_1, B_2, ..., B_K$ is of the form

$$Z = \frac{1}{\sqrt{\sum_{k=1}^{K} a_k^2}} \sum_{k=1}^{K} a_k B_k,$$

where $a_1, a_2, ..., a_K$ are real numbers, or in matrix notation

$$Z = \frac{1}{\sqrt{\mathbf{a}'\mathbf{a}}} \mathbf{a}'\mathbf{B}.$$

We model the price of risky assets as diffusions

$$\frac{dS}{S} = \mu_S(\cdot)dt + \sigma_S(\cdot)dB_S$$

where the drift $\mu(\cdot)$ and the volatility of returns $\sigma(\cdot)$ might depend on time t, uncertainty $\omega \in \Omega$, and potentially other state variables.

The total instantaneous return of an asset is given by

$$\frac{dS + Ddt}{S} = \frac{dS}{S} + \frac{D}{S}dt,$$

where $\frac{D}{S}$ is the dividend yield. The dividend yield of a stock determines the number of new shares that the dividend process generates. Unlike cash dividends, the dividend yield acts as if dividends are reinvested in the stock. Thus, we can always work with dividend-reinvested assets instead. To see this, let

$$\frac{dX}{X} = \frac{D}{S}dt. \tag{1}$$

Here X represents the number of new shares accruing to the owner of the stock determined by the dividend yield. Note that

$$X_t = X_0 \exp\left(\int_0^t \frac{D_u}{S_u}du\right), \tag{2}$$

so the total number of shares grows exponentially with a growth rate equal to the dividend yield of the asset. In other words, X_t keeps track of the total number of shares at each point in time.

The dividend-reinvested asset price is then

$$P = XS,$$

where P denotes the total value of this investment given by the number of shares times the price per share. We can find the dynamics of P by applying Ito's lemma:

$$\frac{dP}{P} = \frac{dS}{S} + \frac{dX}{X} = \frac{dS}{S} + \frac{D}{S}dt.$$

Not surprisingly, the dynamics of P are characterized by capital gains and a dividend yield.

We will assume that there is a risk-free rate of return r. We do not always assume that r is constant, as it could depend on time t, ω or other state variables. We assume that there is a money-market account β that earns a risk-free rate. If we start with β_0 in the account, we must

have that

$$\frac{d\beta}{\beta} = rdt. \tag{3}$$

We can solve for β to find,

$$\beta_t = \beta_0 \exp\left(\int_0^t r_s ds\right). \tag{4}$$

In many applications, the risk-free rate follows a diffusion such that

$$dr = \mu_r(\cdot)dt + \sigma_r(\cdot)dB_r.$$

From Discrete to Continuous Time

We want to derive a stochastic discount factor that works for discounting risky cash flows in continuous time. It will be easier to derive the transition if we can abstract from dividends. However, we cannot just make dividends disappear since the fundamental value of any asset is the present value of all cash flow payments during the life of the asset; otherwise, the asset would be a bubble.

Thus, let's proceed as before and build a dividend-reinvested asset in discrete time. To do this, we will use a dividend yield q that is known at time t and determines how many new shares of the asset we get next period. If the asset price at time 0 is S_0, then at time 1 the asset price will be S_1 and we will have q_0 more shares of the asset, making our total investment worth $P_1 = (1 + q_0)S_1$. If we reinvest the dividends again, at time 2 the value of our investment will be $P_2 = (1 + q_0)(1 + q_1)S_2$. We can continue in this way to find

$$P_{t+1} = X_{t+1}S_{t+1},$$

where we define $X_{t+1} = \prod_{i=0}^t (1 + q_i)$. This definition for X_{t+1} is the discrete-time equivalent of equation (2) in continuous time.

Now, the pricing equation implies that

$$S_t = \mathsf{E}_t \, m_{t+1}(1 + q_t)S_{t+1}.$$

Multiplying both sides by $\prod_{i=0}^{t-1}(1 + q_i)$ we find

$$P_t = \mathsf{E}_t \, m_{t+1} P_{t+1}. \tag{5}$$

Thus, we can do all the math by assuming that we work with a dividend-reinvested asset.

The previous expression is correct in discrete time but it is not the best way to work in continuous time. Assume the stochastic discount factor satisfies $m > 0$ a.s. Then, take $\Lambda_0 > 0$ and define $\Lambda_1 = m_1\Lambda_0, \Lambda_2 = m_2\Lambda_1$, so that $\Lambda_{t+1} = m_{t+1}\Lambda_t > 0$ for all $t \geq 0$.

We can now re-write equation (5) as

$$\Lambda_t P_t = \mathsf{E}_t \, \Lambda_{t+1} P_{t+1}. \tag{6}$$

Before passing to continuous time, note that we can recursively apply the previous expression to P_{t+2}, P_{t+3}, and so forth to find

$$\Lambda_t P_t = \mathsf{E}_t \, \Lambda_T P_T,$$

for any $T = t + n > t$. The process (ΛP) is therefore a martingale. We need to be careful, though, as we do not want the price process P to be a bubble. In terms of the traded asset S we must have that

$$\Lambda_t S_t = \mathsf{E}_t \sum_{i=1}^{n} \Lambda_{t+i} D_{t+i} + \Lambda_{t+n} S_{t+n}.$$

The no-bubbles or transversality condition then implies that

$$\lim_{n\to\infty} \mathsf{E}_t \, \Lambda_{t+n} S_{t+n} = 0.$$

In continuous time we do not have to wait forever for bubbles to appear since in any given time interval we have an infinite number of transactions that can occur. Thus, we will have to be careful about what type of price processes we can admit.

Coming back to passing to continuous time, let's denote the time interval by Δt so that equation (6) becomes

$$\mathsf{E}_t(\Lambda_{t+\Delta} P_{t+\Delta} - \Lambda_t P_t) = 0.$$

If we now let $\Delta t \to 0$, the previous expression implies

$$\mathsf{E}_t \, d(\Lambda_t P_t) = 0.$$

We typically drop the time subscripts when there is no confusion and just write

$$\mathsf{E} \, d(\Lambda P) = 0. \tag{7}$$

The previous expression asserts that the discounted dividend-reinvested price process is a local martingale. We will discuss later situations in which this local martingale is in fact a martingale.

Since $\Lambda > 0$, we have that

$$\begin{aligned}
\frac{d(\Lambda P)}{\Lambda P} &= \frac{dP}{P} + \frac{d\Lambda}{\Lambda} + \frac{d\Lambda}{\Lambda}\frac{dP}{P} \\
&= \frac{dS}{S} + \frac{d\Lambda}{\Lambda} + \frac{d\Lambda}{\Lambda}\frac{dS}{S} + \frac{D}{S}dt \\
&= \frac{d(\Lambda S)}{\Lambda S} + \frac{D}{S}dt.
\end{aligned}$$

Thus, $\mathsf{E} \, d(\Lambda P) = 0$ is equivalent to

$$\mathsf{E} \, d(\Lambda S) + \Lambda D \, dt = 0. \tag{8}$$

Equation (8) is an alternative to equation (7) which makes explicit the dividend process in pricing the asset. Both equations are the continuous-time counterparts to $p = \mathsf{E}(mx)$ in discrete time, which in disguise is also saying that the discounted dividend-reinvested price process is a local martingale.

An SDF in Continuous Time

Let's start computing the discounted process for the money market account β defined earlier. Remember that

$$\frac{d\beta}{\beta} = r \, dt.$$

Thus,

$$d(\Lambda\beta) = \Lambda d\beta + \beta d\Lambda.$$

The pricing equation (8) implies that $E\, d(\Lambda\beta) = 0$, so that

$$E(\Lambda d\beta + \beta d\Lambda) = 0,$$

or

$$E\left(\frac{d\Lambda}{\Lambda}\right) = -E\left(\frac{d\beta}{\beta}\right) = -rdt.$$

Thus, the drift of the SDF in continuous time determines the equilibrium continuously-compounded risk-free rate. Remember that r need not be deterministic but just an adapted process to the filtration of the probability space.

Applying Ito's lemma now to ΛS we find that

$$d(\Lambda S) = Sd\Lambda + \Lambda dS + d\Lambda dS,$$

or

$$\frac{d(\Lambda S)}{\Lambda S} = \frac{d\Lambda}{\Lambda} + \frac{dS}{S} + \frac{d\Lambda}{\Lambda}\frac{dS}{S}.$$

Taking expectations both sides, equation (8) implies that

$$E\left(\frac{dS}{S}\right) + \frac{D}{S}dt = rdt - \left(\frac{d\Lambda}{\Lambda}\right)\left(\frac{dS}{S}\right).$$

Property 1. *Consider an asset S that follows a diffusion*

$$\frac{dS}{S} = \mu dt + \sigma dB.$$

If the asset pays a dividend yield $q = D/S$, and there are no arbitrage opportunities, it must be the case that

$$(\mu + q - r)dt = -\left(\frac{d\Lambda}{\Lambda}\right)\left(\frac{dS}{S}\right). \tag{9}$$

In words, the risk-premium of the asset equals minus the covariance of the SDF and the asset's returns.

Back to Consumption

We can always find the stochastic discount factor from the marginal utility of consumption. Under additive utility, the stochastic discount factor takes the form $\Lambda = e^{-\delta t}u'(c)$. This formulation is useful if we want to understand the link between marginal utility of consumption and discount factors or risk-neutral probabilities.

Applying Ito's lemma to this particular Λ we find

$$d\Lambda = \frac{\partial \Lambda}{\partial c}dc + \frac{1}{2}\frac{\partial^2 \Lambda}{\partial c^2}(dc)^2 + \frac{\partial \Lambda}{\partial t}dt$$

$$= e^{-\delta t}u''(c)dc + \frac{1}{2}e^{-\delta t}u'''(c)(dc)^2 - \delta e^{-\delta t}u'(c)dt,$$

or

$$\frac{d\Lambda}{\Lambda} = -\delta dt + \frac{1}{2}\frac{c^2 u'''(c)}{u'(c)}\left(\frac{dc}{c}\right)^2 + \frac{c u''(c)}{u'(c)}\frac{dc}{c}.$$

For power utility we have that

$$\frac{d\Lambda}{\Lambda} = -\delta dt + \frac{1}{2}\gamma(\gamma+1)\left(\frac{dc}{c}\right)^2 - \gamma \frac{dc}{c}.$$

Let's write

$$\frac{dc}{c} = \mu_c dt + \sigma_c dB_c.$$

Assuming power utility, we have that

$$\frac{d\Lambda}{\Lambda} = \left(-\delta + \frac{1}{2}\gamma(\gamma+1)\sigma_c^2 - \gamma \mu_c\right)dt - \gamma \sigma_c dB_c. \tag{10}$$

We can now recover the risk-free rate dynamics in terms of consumption growth dynamics using

$$r = -\frac{1}{dt}\mathsf{E}\left(\frac{d\Lambda}{\Lambda}\right).$$

The model implies that the instantaneous risk-free rate is given by minus the drift of $d\Lambda/\Lambda$,

$$r = \delta + \gamma\mu_c - \frac{1}{2}\gamma(\gamma + 1)\sigma_c^2.$$

The expression is very intuitive and has important implications. First, real interest rates are high when impatience (δ) is high since more impatient investors will require a high interest rate to save. Furthermore, interest rates are high when expected consumption growth (μ_c) is high. Indeed, if agents expect consumption to go up, they need to save less, pushing bond prices down. Finally, interest rates are high when volatility of future consumption growth (σ_c) is low. This phenomenon is usually called **precautionary savings**. If agents are less afraid of future consumption growth, they bid bond prices down pushing interest rates up.

We can also use this consumption-based asset pricing model to understand what is called the **equity premium puzzle**. Consider an asset paying a dividend flow Ddt and following a diffusion

$$\frac{dS}{S} = \mu_S dt + \sigma_S dB_S$$

such that $(dB_S)(dB_c) = \rho dt$. Equations (9) and (10) imply

$$\mu_S + D/S - r = \gamma\rho\sigma_c\sigma_S.$$

In this simple asset pricing model with power utility, the risk premium of any risky asset is higher when risk aversion (γ) is high and/or the covariance of asset returns and consumption growth is high.

Since $|\rho| \le 1$, the previous expression implies

$$\left|\frac{\mu_S + D/S - r}{\sigma_S}\right| \le \gamma\sigma_c.$$

In the data, the Sharpe ratio of the market is around 0.5 whereas the standard deviation of consumption growth is around 0.01. We need a RRA coefficient of at least 50 to explain the risk-premium of the market! To solve this paradox, researchers have introduced preferences that generate a more volatile stochastic discount factor, such as recursive Epstein-Zin preferences or habits.

Generic SDFs in Continuous Time

There are $N \leq K$ securities whose price process follow a diffusion

$$\frac{dS_i}{S_i} = \mu_i dt + \boldsymbol{\sigma}_i d\mathbf{B}, \tag{11}$$

where $\boldsymbol{\sigma}_i$ is a $K \times 1$ vector. Each security pays continuously a dividend yield $q_i dt$.

Define

$$\frac{d\mathbf{S}}{\mathbf{S}} = \left(\frac{dS_1}{S_1} \quad \frac{dS_2}{S_2} \quad \cdots \quad \frac{dS_N}{S_N} \right)'$$

and denote by $\boldsymbol{\sigma}$ the $N \times K$ matrix whose rows are given by σ_i defined in (11). We have that

$$\frac{d\mathbf{S}}{\mathbf{S}} = \boldsymbol{\mu} dt + \boldsymbol{\sigma} d\mathbf{B},$$

implying

$$\left(\frac{d\mathbf{S}}{\mathbf{S}} \right) \left(\frac{d\mathbf{S}}{\mathbf{S}} \right)' = \boldsymbol{\sigma}\boldsymbol{\sigma}' dt.$$

The $N \times N$ matrix $\boldsymbol{\sigma}\boldsymbol{\sigma}'$ determines the instantaneous covariance of returns. The SDF

$$\frac{d\Lambda}{\Lambda} = -r\, dt - \left(\boldsymbol{\mu} + \mathbf{q} - r\boldsymbol{\iota} \right)' \left(\boldsymbol{\sigma}\boldsymbol{\sigma}' \right)^{-1} \boldsymbol{\sigma}\, d\mathbf{B} \tag{12}$$

prices all N assets correctly. For asset i, the pricing equation (9) requires $(\mu_i + q_i - r)\, dt = -\frac{d\Lambda}{\Lambda} \frac{dS_i}{S_i}$. Using $(d\mathbf{B})(d\mathbf{B})' = \mathbf{I}\, dt$, the right-hand side equals $(\boldsymbol{\mu} + \mathbf{q} - r\boldsymbol{\iota})'(\boldsymbol{\sigma}\boldsymbol{\sigma}')^{-1}\boldsymbol{\sigma}\boldsymbol{\sigma}_i\, dt$. Since $\boldsymbol{\sigma}\boldsymbol{\sigma}_i$ is the i-th column of $\boldsymbol{\sigma}\boldsymbol{\sigma}'$, the inverse cancels and we recover $\mu_i + q_i - r$, as required.

Here $\mathbf{q}$ denotes the vector of dividend yields

$$\mathbf{q} = \left(\frac{D_1}{S_1}, \frac{D_2}{S_2}, \cdots, \frac{D_N}{S_N} \right)'.$$

The Intertemporal CAPM

Let $V(W, \mathbf{z})$ denote the investor's value function, where W is wealth and $\mathbf{z}$ is a vector of state variables that shift the investment opportunity set. At an optimum, the marginal utility of consumption equals the marginal value of wealth,

$$u'(c) = V_W,$$

so the SDF can be written as

$$\Lambda = e^{-\delta t} V_W(W, \mathbf{z}).$$

Applying Ito's lemma to Λ, the stochastic component is

$$\frac{d\Lambda}{\Lambda} = (\cdot)\, dt + \frac{W V_{WW}(W, \mathbf{z})}{V_W(W, \mathbf{z})} \frac{dW}{W} + \frac{V_{W\mathbf{z}'}(W, \mathbf{z})}{V_W(W, \mathbf{z})} d\mathbf{z}. \tag{13}$$

Substituting into the fundamental pricing equation (9) yields Merton's Intertemporal CAPM.

Property 2 (Merton's Intertemporal CAPM). *Let $rra = -W V_{WW}(W, \mathbf{z})/V_W(W, \mathbf{z}) > 0$ be the coefficient of relative risk aversion. The equilibrium risk premium of any risky asset satisfies*

$$\mathsf{E}\left(\frac{dS}{S}\right) + \frac{D}{S} dt - r\, dt = rra \cdot \frac{dW}{W} \frac{dS}{S} - \frac{V_{W\mathbf{z}'}(W, \mathbf{z})}{V_W(W, \mathbf{z})} \left(d\mathbf{z} \frac{dS}{S}\right). \tag{14}$$

Expected excess returns depend on $K + 1$ factors: the wealth portfolio and K state-variable hedging portfolios, one for each source of time-variation in the investment opportunity set.

The risk premium has two economically distinct components:

1. **Wealth risk premium.** The term $rra \cdot (dW/W)(dS/S)/dt$ is the instantaneous covariance of asset returns with wealth growth, scaled by the coefficient of relative risk aversion. Assets that covary positively with aggregate wealth are risky and command higher expected returns.

2. **Hedging demand premium.** The term $-(V_{W\mathbf{z}'}/V_W)(d\mathbf{z})(dS/S)/dt$ reflects investors' desire to hedge against shifts in the investment opportunity set. An asset that covaries positively with a state variable z_k for which $V_{Wz_k} > 0$ acts as a hedge against deteriorating investment opportunities and therefore commands a lower risk premium.

When there are no state variables — or whenever $V_{Wz_k} = 0$ for all k — the ICAPM reduces to a single-factor model in which the market (wealth) portfolio is the only priced risk. In that special case, equation (14) recovers the continuous-time CAPM: risk premiums are proportional to covariance with aggregate wealth.

The Market Price of Risk

Any continuous-time SDF can be written as

$$\frac{d\Lambda}{\Lambda} = -r\,dt - '\,d\mathbf{B}, \tag{15}$$

where $= (\lambda_1, \lambda_2, \dots, \lambda_K)' \in \mathbb{R}^K$ is the **market price of risk** vector. The k-th component λ_k measures the excess instantaneous return per unit of exposure to the k-th Brownian shock B_k.

From the fundamental pricing equation (9), the risk premium of asset i satisfies

$$(\mu_i + q_i - r)\,dt = -\frac{d\Lambda}{\Lambda}\frac{dS_i}{S_i} = '(d\mathbf{B})(d\mathbf{B})'\boldsymbol{\sigma}_i = \boldsymbol{\sigma}_i'\,dt, \tag{16}$$

using $(d\mathbf{B})(d\mathbf{B})' = \mathbf{I}\,dt$. Stacking all N assets in vector form:

$$\boldsymbol{\mu} + \mathbf{q} - r\boldsymbol{\iota} = \boldsymbol{\sigma}. \tag{17}$$

Each asset's risk premium equals its diffusion vector $\boldsymbol{\sigma}_i$ dotted with the market price of risk .

Incomplete markets. In general $N \leq K$, so (17) is a system of N equations in K unknowns and has infinitely many solutions. The SDF formula (12) corresponds to the **minimum-norm** market price of risk vector,

$$_{\min} = \boldsymbol{\sigma}'\left(\boldsymbol{\sigma}\boldsymbol{\sigma}'\right)^{-1}\left(\boldsymbol{\mu} + \mathbf{q} - r\boldsymbol{\iota}\right). \tag{18}$$

Any other solution takes the form $= _{\min} + $, where $\boldsymbol{\sigma} = \mathbf{0}$, i.e., lies in the null space of $\boldsymbol{\sigma}$. Adding alters the SDF but leaves all N risk premiums unchanged, reflecting the non-uniqueness of the SDF in incomplete markets.

The continuous-time Hansen-Jagannathan bound. The instantaneous return variance of asset i is $(\boldsymbol{\sigma}_i' d\mathbf{B})^2/dt = \boldsymbol{\sigma}_i'\boldsymbol{\sigma}_i = \sigma_i^2$, where $\sigma_i \equiv \|\boldsymbol{\sigma}_i\| > 0$ is the scalar instantaneous return volatility. Using (16) and the Cauchy-Schwarz inequality, $|\mu_i + q_i - r| = |\boldsymbol{\sigma}_i'| \leq \sigma_i \|\|$, so dividing by σ_i gives

$$\left| \frac{\mu_i + q_i - r}{\sigma_i} \right| \leq \|\|. \tag{19}$$

The norm $\|\|$ is an upper bound on the instantaneous Sharpe ratio of every risky asset. In the data, U.S. equity has delivered a Sharpe ratio of roughly 0.5 per year, which requires $\|\| \geq 0.5$ for any admissible SDF. Equality in (19) holds for an asset whose diffusion vector is proportional to , i.e., an asset perfectly negatively correlated with the SDF. This is the continuous-time counterpart of the Hansen-Jagannathan bound.

The Risk-Neutral Measure

The SDF Λ and the money-market account β together define the **pricing kernel deflator**

$$\mathcal{E}_t = \frac{\Lambda_t \beta_t}{\Lambda_0 \beta_0}.$$

Using (15) and (3),

$$\frac{d\mathcal{E}}{\mathcal{E}} = \frac{d\Lambda}{\Lambda} + \frac{d\beta}{\beta} = -'d\mathbf{B},$$

so $\mathcal{E}$ is a strictly positive local martingale with $\mathcal{E}_0 = 1$. When $\mathcal{E}$ is a true martingale — Novikov's condition

$$\mathsf{E}\left[\exp\left(\frac{1}{2} \int_0^T \|_t\|^2 \, dt \right) \right] < \infty$$

is a standard sufficient criterion — we define the **risk-neutral measure** P^* on $\mathcal{F}_T$ by

$$\frac{d\mathsf{P}^*}{d\mathsf{P}} = \mathcal{E}_T.$$

Girsanov's theorem. Under P^*, the process

$$d\mathbf{B}^* = d\mathbf{B} + \, dt$$

is a K-dimensional standard Brownian motion. The P^*-dynamics of asset i then follow from substituting back:

$$\frac{dS_i}{S_i} = \mu_i \, dt + \boldsymbol{\sigma}_i' d\mathbf{B}$$
$$= \left(\mu_i - \boldsymbol{\sigma}_i'\right) dt + \boldsymbol{\sigma}_i' d\mathbf{B}^*$$
$$= (r - q_i) \, dt + \boldsymbol{\sigma}_i' d\mathbf{B}^*,$$

where the last step uses (16). Under P^*, every asset's expected total return (capital gain plus dividend yield) equals the risk-free rate:

$$\mathsf{E}^* \left(\frac{dS_i + D_i \, dt}{S_i}\right) = r \, dt.$$

In a risk-neutral world, all investors are content discounting every cash flow at the risk-free rate, regardless of its risk profile.

Risk-neutral pricing formula. Since $\mathcal{E}$ is a P-martingale, $\Lambda\beta$ is also a martingale, and for any $\mathcal{F}_T$-measurable payoff V_T:

$$V_t = \mathsf{E}_t^* \left[e^{-\int_t^T r_s \, ds} V_T\right]. \tag{20}$$

All cash flows are discounted at the risk-free rate and expectations are taken under P^*. This representation is the foundation of continuous-time derivatives pricing; see Derivatives Pricing in Continuous Time for applications to options and forward contracts.

Example 1 (Zero-Coupon Bonds). A zero-coupon bond maturing at time T pays 1 unit of consumption at maturity. Its time-t price is

$$B_t(T) = \mathsf{E}_t^* \left[e^{-\int_t^T r_s \, ds}\right].$$

When the short rate is constant, $B_t(T) = e^{-r(T-t)}$, recovering the standard continuously-compounded discount factor. With a stochastic short rate, $B_t(T)$ depends on the distribution of future short rates under P^*, which is the starting point for term-structure modeling. □

Connection between the SDF and the risk-neutral measure. The two representations are equivalent: given P^* we can recover the SDF as $\Lambda_t = \mathcal{E}_t/\beta_t$ (up to normalization), and vice versa. In complete markets ($N = K$ and $\boldsymbol{\sigma}$ invertible), is unique and so is P^*. In incomplete

markets, there are infinitely many risk-neutral measures — one for each choice of in the null space of $\boldsymbol{\sigma}$ — reflecting the fundamental non-uniqueness of pricing in markets with unspanned risk.