Lognormal Consumption Growth

Lorenzo Naranjo

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This notebook develops the lognormal benchmark for asset pricing, starting with CRRA power utility and extending to Epstein-Zin recursive preferences. The lognormal setting is the workhorse framework for theoretical asset pricing because it yields closed-form expressions for the risk-free rate, the Hansen-Jagannathan bound, and log risk premia—all reducing to simple functions of means, variances, and covariances. This tractability follows from a single key property: when the log SDF and log returns are jointly normal, the pricing equation $1 = \mathsf{E}_t(M_{t+1}R_{i,t+1})$ becomes an equation in moments rather than an integral. Risk premia then decompose into covariance terms with transparent economic interpretations, and comparative statics on preference parameters follow directly.

The lognormal framework is also the natural setting for diagnosing the two central puzzles in consumption-based asset pricing. Under CRRA preferences, matching the observed equity premium requires implausibly high risk aversion, which in turn implies a counterfactually high risk-free rate—the equity premium and risk-free rate puzzles arise precisely because a single parameter governs both. Epstein-Zin preferences resolve this within the same lognormal environment by decoupling risk aversion from the elasticity of intertemporal substitution, showing how the two puzzles can be addressed independently without abandoning the analytical convenience of the framework. This tractability also underpins the asset pricing results developed in Risk Premia in Intertemporal Models, where the lognormal pricing equations derived here serve as the point of departure for the Campbell and Bansal-Yaron frameworks.

Let lowercase letters denote logs, so $c_t = \ln(C_t)$ and $\Delta c_{t+1} = c_{t+1} - c_t$. Throughout, assume consumption growth is conditionally normal:

$$\Delta c_{t+1} \sim \mathcal{N}\left(\mathsf{E}_t(\Delta c_{t+1}), \sigma_t^2(\Delta c_{t+1})\right).$$

This assumption, together with joint normality of log returns and the log SDF, underlies all closed-form results derived below.

Power Utility

As a benchmark before introducing Epstein-Zin recursion, start with the standard power-utility model. This baseline pins down how interest rates and risk premia relate to marginal utility when risk aversion and intertemporal substitution are tied together. Preferences are CRRA (power utility):

$$u(C) = \begin{cases} \frac{C^{1-\gamma}}{1-\gamma}, & \text{if } \gamma \geq 0, \gamma \neq 1 \\ \ln(C), & \text{if } \gamma = 1 \end{cases}$$

with marginal utility $u'(C) = C^{-\gamma}$.

With $\delta = -\ln(\beta)$, the log SDF is

$$m_{t+1} = -\delta - \gamma \Delta c_{t+1}.$$

Since m_{t+1} is an affine function of the conditionally normal Δc_{t+1}, it is also conditionally normal with

$$\begin{aligned} \mathrm{E}_t(m_{t+1}) &= -\delta - \gamma\, \mathrm{E}_t(\Delta c_{t+1}), \\ \sigma_t^2(m_{t+1}) &= \gamma^2 \sigma_t^2(\Delta c_{t+1}). \end{aligned}$$

The Risk-Free Rate

Since $M_{t+1} = e^{m_{t+1}}$ is lognormal, applying the lognormal expectation formula gives

$$\begin{aligned} \mathrm{E}_t(M_{t+1}) &= \exp\left(\mathrm{E}_t(m_{t+1}) + \frac{1}{2}\sigma_t^2(m_{t+1})\right) \\ &= \exp\left(-\delta - \gamma\, \mathrm{E}_t(\Delta c_{t+1}) + \frac{1}{2}\gamma^2 \sigma_t^2(\Delta c_{t+1})\right). \end{aligned}$$

Using $r_{f,t+1} = -\ln \mathrm{E}_t(M_{t+1})$, the log risk-free rate is

$$r_{f,t+1} = \delta + \gamma\, \mathrm{E}_t(\Delta c_{t+1}) - \frac{1}{2}\gamma^2 \sigma_t^2(\Delta c_{t+1}).$$

This expression implies three comparative statics for the real risk-free rate. Higher impatience

(δ) raises the risk-free rate. Higher expected consumption growth also raises the risk-free rate because if households expect to be richer tomorrow, they save less today, lowering bond demand and raising yields. By contrast, higher consumption-growth uncertainty lowers the risk-free rate through precautionary saving. In the data, the real risk-free rate is low and stable despite positive expected consumption growth—a tension sometimes called the *risk-free rate puzzle*—which Epstein-Zin preferences help address by decoupling the precautionary-saving response from risk aversion.

The Equity Premium Puzzle

The Hansen-Jagannathan bound states that the maximum Sharpe ratio achievable by any asset equals $\sigma_t(M_{t+1})/\,\mathrm{E}_t(M_{t+1})$. For a lognormal SDF, the lognormal moment formulas give $\sigma(M)/\,\mathrm{E}(M) = \sqrt{e^{\bar{\sigma}^2} - 1} \approx \bar{\sigma}$, so the bound on the market Sharpe ratio becomes

$$\left| \frac{\mathrm{E}_t(R_{m,t+1}) - R_{f,t+1}}{\sigma_t(R_{m,t+1})} \right| \leq \gamma \sigma_t(\Delta c_{t+1}).$$

In the data, the market Sharpe ratio is around 0.5, whereas the standard deviation of consumption growth is around 0.01, implying an RRA coefficient of at least 50.

These two results are jointly inconsistent with the data. Matching the observed Sharpe ratio requires $\gamma \geq 50$, but substituting such a value into the risk-free rate formula gives $r_{f,t+1} \approx \delta + 50 \cdot \mathrm{E}_t(\Delta c_{t+1}) - \frac{1}{2} \cdot 2500 \cdot \sigma_t^2(\Delta c_{t+1})$, which is far above observed real interest rates for any plausible calibration of consumption growth moments. The root cause is that a single parameter γ simultaneously governs risk aversion and the intertemporal elasticity of saving. Epstein-Zin preferences resolve this by decoupling the two, allowing the equity premium and the risk-free rate to be addressed independently.

Log Risk Premium

With m_{t+1} and $r_{i,t+1}$ jointly conditionally normal, the pricing equation expands as

$$\begin{aligned} 1 &= \mathsf{E}_t(M_{t+1}R_{i,t+1}) \\ &= \mathsf{E}_t(e^{m_{t+1}+r_{i,t+1}}) \\ &= e^{\mathsf{E}_t(m_{t+1}+r_{i,t+1})+\frac{1}{2}\sigma_t^2(m_{t+1}+r_{i,t+1})}, \end{aligned}$$

which requires

$$\mathsf{E}_t(m_{t+1}+r_{i,t+1}) + \frac{1}{2}\sigma_t^2(m_{t+1}+r_{i,t+1}) = 0. \tag{1}$$

Equation (1) implies that

$$\mathsf{E}_t(r_{i,t+1}) - r_{f,t+1} + \frac{V_{ii,t}}{2} = \gamma V_{ic,t} \tag{2}$$

where $V_{ii,t} = \sigma_t^2(r_{i,t+1})$ and $V_{ic,t} = \mathsf{Cov}_t(\Delta c_{t+1}, r_{i,t+1})$. To see why, apply (1) separately to the risk-free asset—whose return $r_{f,t+1}$ is known at t, so $\sigma_t^2(m_{t+1}+r_{f,t+1}) = \sigma_t^2(m_{t+1})$—and subtract to get

$$\mathsf{E}_t(r_{i,t+1}) - r_{f,t+1} + \frac{V_{ii,t}}{2} = -\,\mathsf{Cov}_t(m_{t+1}, r_{i,t+1}) = \gamma V_{ic,t}.$$

Equation (2) says the expected excess log return is increasing in an asset's covariance with consumption growth. Assets that pay off when consumption is already high provide poor insurance and therefore require higher expected returns. The term $V_{ii,t}/2$ is a Jensen's inequality correction intrinsic to log returns and is negligible in practice.

Epstein-Zin Preferences

The standard CRRA time-additive model is equivalent to a recursive specification: the continuation value U_t satisfies the Bellman equation

$$U_t^{1-\gamma} = (1-\beta)C_t^{1-\gamma} + \beta\,\mathsf{E}_t(U_{t+1}^{1-\gamma}).$$

In this formulation, the agent's attitude toward risk and their willingness to substitute consumption over time are both governed by the single parameter γ. Specifically, γ is the coefficient of relative risk aversion, while the elasticity of intertemporal substitution (EIS) is constrained to equal $1/\gamma$. This tight link is a restrictive feature of the standard model: an agent who is very risk averse (large γ) is also forced to have a very low willingness to substitute consumption over time, even though these are conceptually distinct aspects of preferences.

The Epstein-Zin model decouples these two parameters by introducing an additional scaling parameter θ into the recursion:

$$U_t^{\frac{1-\gamma}{\theta}} = (1-\beta)C_t^{\frac{1-\gamma}{\theta}} + \beta\left(\mathsf{E}_t(U_{t+1}^{1-\gamma})\right)^{\frac{1}{\theta}},$$

where

$$\theta = \frac{1-\gamma}{1-1/\psi}.$$

The parameter ψ now independently controls the EIS, so γ and ψ can be chosen freely to match empirical estimates of risk aversion and intertemporal substitution separately. When $\psi = 1/\gamma$ we have $\theta = 1$ and the standard power utility model is recovered as a special case. Intuitively, θ governs the relative weight on intertemporal substitution versus market-return news in the SDF: when $\theta < 1$ (equivalently, $\gamma > 1/\psi$ for $\psi > 1$) the agent prefers early resolution of uncertainty, and this tilts the SDF toward the market-return factor.

The Budget Constraint

As established in Recursive Preferences in a Multiperiod Economy, wealth evolves according to

$$W_{t+1} = R_{w,t+1}(W_t - C_t) \tag{3}$$

where $R_{w,t+1}$ is the gross return on the market portfolio — a claim to the agent's entire future consumption stream.

The Stochastic Discount Factor

As derived in Recursive Preferences in a Multiperiod Economy, the stochastic discount factor takes the form

$$M_{t+1} = \left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\right]^{\theta}\left[\frac{1}{R_{w,t+1}}\right]^{1-\theta}.$$

This expression makes the separation of preferences transparent. The first factor is an intertemporal substitution term: it depends on consumption growth raised to $-1/\psi$, the inverse of the EIS. The second factor captures the return on the market portfolio and reflects how utility is affected by revisions in future investment opportunities. In the standard power utility case ($\theta = 1$), the second factor vanishes and only consumption growth determines the SDF.

Taking logs, with $\delta = -\ln\beta$ and lowercase letters denoting logs of their uppercase counterparts, the log SDF is

$$m_{t+1} = -\theta\delta - (\theta/\psi)\Delta c_{t+1} - (1-\theta)r_{w,t+1}.$$

The Risk-Free Rate

Applying (1) to the risk-free asset gives $r_{f,t+1} = -\operatorname{E}_t(m_{t+1}) - \frac{1}{2}\sigma_t^2(m_{t+1})$, so

$$r_{f,t+1} = \theta\delta + \frac{\theta}{\psi}\operatorname{E}_t(\Delta c_{t+1}) + (1-\theta)\operatorname{E}_t(r_{w,t+1}) - \frac{1}{2}\sigma_t^2(m_{t+1}),$$

where $\sigma_t^2(m_{t+1}) = (\theta/\psi)^2 V_{cc,t} + (1-\theta)^2 V_{ww,t} + 2(\theta/\psi)(1-\theta)V_{cw,t}$, with $V_{ww,t} = \sigma_t^2(r_{w,t+1})$ and $V_{cw,t} = \operatorname{Cov}_t(\Delta c_{t+1}, r_{w,t+1})$. Setting $\theta = 1$ and $\psi = 1/\gamma$ recovers the CRRA expression.

The decoupling is immediate: expected consumption growth enters with coefficient θ/ψ and the precautionary saving term scales with $(\theta/\psi)^2$, both of which depend on ψ independently of γ. A high γ can therefore match the equity premium without mechanically inflating the risk-free rate, resolving the tension identified in the Power Utility section.

Log Risk Premium

Using (1) and substituting the EZ log SDF into $-\operatorname{Cov}_t(m_{t+1}, r_{i,t+1})$, the log risk premium for any asset i satisfies

$$\mathrm{E}_t(r_{i,t+1}) - r_{f,t+1} + \frac{V_{ii,t}}{2} = \frac{\theta}{\psi} V_{ic,t} + (1-\theta) V_{iw,t} \tag{4}$$

where $V_{ic,t} = \operatorname{Cov}_t(r_{i,t+1}, \Delta c_{t+1})$ and $V_{iw,t} = \operatorname{Cov}_t(r_{i,t+1}, r_{w,t+1})$. The left-hand side is the expected excess log return plus a convexity correction. The right-hand side loads on two covariances—with consumption growth and with the market return—with weights that depend on θ and hence on the preference parameters γ and ψ.

When $\psi = 1/\gamma$ we have $\theta = 1$, so $\theta/\psi = \gamma$ and $1 - \theta = 0$. Equation (4) then reduces to (2): risk is priced solely through consumption covariance at rate γ, and the market return plays no role.

Outside this special case the model is an intertemporal CAPM with two priced betas. The first, $V_{ic,t}$, captures short-run consumption risk—the same margin as CRRA. The second, $V_{iw,t}$, captures exposure to the market portfolio and reflects intertemporal hedging demand. Its weight $1-\theta$ depends on how far preferences deviate from CRRA. When $\psi > 1$ and $\gamma > 1/\psi$—the empirically relevant case in which the agent prefers early resolution of uncertainty—we have $1 - \theta > 1$, so market-return covariance is priced more heavily than under CRRA. An asset that co-moves strongly with the market amplifies wealth fluctuations precisely when continuation utility is sensitive to them, making it riskier than its consumption beta alone would suggest.