Recursive Preferences in a Multiperiod Economy

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Introduction

The Epstein-Zin preferences notebook derived the SDF in a two-date setting by guessing a homothetic consumption rule and solving a single maximization problem. That approach is clean but sidesteps the recursive structure that makes Epstein-Zin preferences genuinely useful in multiperiod environments. This notebook derives the same SDF directly from the multiperiod recursive utility aggregator, relying on three tools: implicit differentiation of the recursive equation, a perturbation argument to extract the pricing kernel, and Euler's homogeneous function theorem to translate utility into observable quantities.

The payoff is the standard Epstein-Zin stochastic discount factor

$$m_{t+1} = \left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-1/\psi}\right]^{\theta}\left[\frac{1}{R^w_{t+1}}\right]^{1-\theta}, \qquad \theta = \frac{1-\gamma}{1-1/\psi},$$

expressed entirely in terms of consumption growth and the return on the wealth portfolio. The parameter θ controls the weight of each factor: when $\psi = 1/\gamma$, $\theta = 1$ and the formula collapses to the standard CRRA stochastic discount factor; otherwise the wealth return enters as a separate pricing factor, capturing the agent's concern for future investment opportunities — the source of Epstein-Zin's intertemporal hedging demand. The multiperiod derivation confirms that the two-date formula is exact in every period and not merely an approximation valid near a steady state.

General Recursive Utility

Kreps and Porteus (1978) develop a framework for recursive utility that separates attitudes toward risk from the willingness to substitute consumption over time. A general Kreps–Porteus aggregator takes the form

$$V_t = U(c_t, \mu_t),$$

where $U(c, \mu)$ is an increasing aggregator and μ_t is the certainty equivalent of next-period utility under a von Neumann–Morgenstern risk aggregator:

$$\mu_t = \phi^{-1}(\mathsf{E}_t[\phi(V_{t+1})]),$$

for some increasing, concave ϕ. The concavity of ϕ makes the agent averse to uncertainty about future utility levels, while the curvature of U in its first argument separately controls the willingness to substitute consumption across time. This is the key departure from standard expected utility, where a single function governs both simultaneously. Epstein-Zin preferences (Epstein and Zin 1989) are the leading special case, with $U(c, \mu) = [(1-\beta)c^\rho + \beta\mu^\rho]^{1/\rho}$ and $\phi(v) = v^{1-\gamma}$.

To derive the SDF, consider a small perturbation in the consumption plan: give up $\xi\, p_t$ units of consumption today in exchange for $\xi\, x_{t+1}$ units tomorrow. At the optimum the agent is indifferent at the margin, so the derivative of V_t with respect to ξ vanishes at $\xi = 0$. Differentiating $U(c_t - \xi p_t,\, \mu_t(\xi))$ at $\xi = 0$ and using $\partial V_{t+1}/\partial c_{t+1} = U_c(c_{t+1}, \mu_{t+1})$ recursively gives the first-order condition

$$p_t\, U_c(c_t, \mu_t) = U_\mu(c_t, \mu_t)\, \frac{\mathsf{E}_t[\phi'(V_{t+1})\, U_c(c_{t+1}, \mu_{t+1})\, x_{t+1}]}{\phi'(\mu_t)},$$

so the SDF is

$$m_{t+1} = \underbrace{\frac{U_\mu(c_t, \mu_t)}{U_c(c_t, \mu_t)}}_{\text{aggregator ratio}} \cdot \underbrace{\frac{\phi'(V_{t+1})}{\phi'(\mu_t)}}_{\text{risk adjustment}} \cdot \underbrace{U_c(c_{t+1}, \mu_{t+1})}_{\text{marginal utility}}. \tag{1}$$

The middle factor $\phi'(V_{t+1})/\phi'(\mu_t)$ compares realized utility to the certainty equivalent through the lens of ϕ. For power $\phi(v) = v^{1-\gamma}$ this is $(V_{t+1}/\mu_t)^{-\gamma}$; for entropic $\phi(v) = -e^{-\alpha v}$ it becomes $e^{-\alpha(V_{t+1}-\mu_t)}$.

Epstein-Zin Preferences

The agent has preferences over consumption streams $(c_t, c_{t+1}, c_{t+2}, \ldots)$ of the Kreps–Porteus form $V_t = U(c_t, \mu_t)$, with aggregator $U(c, \mu) = [(1-\beta)c^\rho + \beta\mu^\rho]^{1/\rho}$ and power risk aggregator $\phi(v) = v^{1-\gamma}$, so the certainty equivalent is

$$\mu_t = \left[\mathsf{E}_t\left(V_{t+1}^{1-\gamma}\right)\right]^{1/(1-\gamma)}.$$

The parameters $\beta \in (0,1)$, $\gamma > 0$, and $\psi > 0$ are the time-discount factor, coefficient of relative risk aversion, and elasticity of intertemporal substitution. Defining $\rho = 1 - 1/\psi$, the aggregator takes the explicit form

$$V_t = \left[(1-\beta)\, c_t^\rho + \beta\, \mu_t^\rho\right]^{1/\rho}, \tag{2}$$

or equivalently,

$$V_t^\rho = (1-\beta)\, c_t^\rho + \beta\, \mu_t^\rho. \tag{3}$$

When $\psi = 1/\gamma$ the aggregator collapses to standard CRRA utility with coefficient γ.

Homotheticity and the Wealth Identity

The aggregator (2) is linearly homogeneous in consumption: scaling every current and future consumption level by $\lambda > 0$ scales V_t by λ. Euler's theorem therefore gives

$$V_t = \sum_{\tau=0}^{\infty} \frac{\partial V_t}{\partial c_{t+\tau}}\, c_{t+\tau} = \mathsf{E}_t\left[\sum_{\tau=0}^{\infty} \frac{\partial V_t}{\partial c_{t+\tau}}\, c_{t+\tau}\right],$$

where the second equality holds because the sum equals V_t state by state, hence in conditional expectation as well. Dividing by $U_c(c_t, \mu_t)$, the right-hand side is the present value of the entire consumption stream — total wealth W_t:

$$\frac{V_t}{U_c(c_t, \mu_t)} = \mathsf{E}_t\left[\sum_{\tau=0}^{\infty} \frac{U_c(c_{t+\tau}, \mu_{t+\tau})}{U_c(c_t, \mu_t)}\, c_{t+\tau}\right] = W_t. \tag{4}$$

Substituting $U_c(c_t, \mu_t) = (1-\beta)\, c_t^{-1/\psi}\, V_t^{1/\psi}$ into $V_t = U_c(c_t, \mu_t)\, W_t$ and simplifying:

$$V_t = (1-\beta)\, c_t^{-1/\psi}\, V_t^{1/\psi}\, W_t \implies V_t^{\rho} = (1-\beta)\, c_t^{-1/\psi}\, W_t, \tag{5}$$

so that

$$V_t = \left[(1-\beta)\, c_t^{-1/\psi}\, W_t\right]^{1/\rho}. \tag{6}$$

The entire infinite-dimensional consumption stream is thus compressed into two quantities: current consumption c_t and total wealth W_t.

The Certainty Equivalent

Substituting (5) into the recursive equation (3) gives

$$(1-\beta)\, c_t^{-1/\psi}\, W_t = (1-\beta)\, c_t^{\rho} + \beta\, \mu_t^{\rho}.$$

Writing $(1-\beta)c_t^{\rho} = (1-\beta)c_t^{-1/\psi} \cdot c_t$ and rearranging:

$$(1-\beta)\, c_t^{-1/\psi}(W_t - c_t) = \beta\, \mu_t^{\rho}.$$

Here $W_t - c_t$ is the amount saved after current consumption. Those savings are invested and grow to next-period wealth: $W_{t+1} = R_{t+1}^{w}\,(W_t - c_t)$, where R_{t+1}^{w} is the gross return on the wealth portfolio. Solving for the certainty equivalent:

$$\mu_t = \left[\frac{(1-\beta)\, c_t^{-1/\psi}(W_t - c_t)}{\beta}\right]^{1/\rho}. \tag{7}$$

Deriving the Pricing Kernel

We apply (1) to the Epstein-Zin aggregator $U(c,\mu) = [(1-\beta)c^{\rho} + \beta\mu^{\rho}]^{1/\rho}$ with risk aggregator $\phi(v) = v^{1-\gamma}$, for which $\mu_t = [\mathrm{E}_t(V_{t+1}^{1-\gamma})]^{1/(1-\gamma)}$. The partial derivatives of U are

$$U_c = (1-\beta)\, c^{\rho-1}\, U^{1-\rho}, \qquad U_\mu = \beta\, \mu^{\rho-1}\, U^{1-\rho},$$

so their ratio is

$$\frac{U_\mu}{U_c} = \frac{\beta\, \mu^{\rho-1}}{(1-\beta)\, c^{\rho-1}}.$$

Since $\phi'(v) = (1-\gamma)v^{-\gamma}$, the risk-adjustment factor reduces to

$$\frac{\phi'(V_{t+1})}{\phi'(\mu_t)} = \left(\frac{V_{t+1}}{\mu_t}\right)^{-\gamma}.$$

Substituting into (1) and collecting terms using $\rho - 1 = -1/\psi$:

$$m_{t+1} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-1/\psi} \mu_t^{\rho-1+\gamma}\, V_{t+1}^{1/\psi-\gamma} = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-1/\psi} \left(\frac{V_{t+1}}{\mu_t}\right)^{1/\psi-\gamma}. \tag{8}$$

This is an exact expression for the SDF in terms of the utility process; (6) and (7) now translate the ratio V_{t+1}/μ_t into observables:

$$\frac{V_{t+1}}{\mu_t} = \frac{\left[(1-\beta)\, c_{t+1}^{-1/\psi}\, W_{t+1}\right]^{1/\rho}}{\left[\dfrac{(1-\beta)\, c_t^{-1/\psi}(W_t - c_t)}{\beta}\right]^{1/\rho}} = \left[\beta\, \frac{W_{t+1}}{W_t - c_t} \left(\frac{c_{t+1}}{c_t}\right)^{-1/\psi}\right]^{1/\rho} = \left[\beta\, R_{t+1}^{w} \left(\frac{c_{t+1}}{c_t}\right)^{-1/\psi}\right]^{1/\rho},$$

where the last step uses $W_{t+1}/(W_t - c_t) = R_{t+1}^w$. Substituting into (8) and collecting terms:

$$m_{t+1} = \beta^{(1-\gamma)/\rho} \left(R_{t+1}^w\right)^{(1/\psi-\gamma)/\rho} \left(\frac{c_{t+1}}{c_t}\right)^{-(1-\gamma)/(\rho\psi)}.$$

Defining $\theta = (1-\gamma)/\rho = (1-\gamma)/(1-1/\psi)$, so that $1-\theta = (\gamma - 1/\psi)/\rho$, the Epstein-Zin stochastic discount factor is

$$m_{t+1} = \beta^{\theta} \left(\frac{c_{t+1}}{c_t}\right)^{-\theta/\psi} (R_{t+1}^w)^{\theta-1}, \tag{9}$$

or equivalently,

$$m_{t+1} = \left[\beta\left(\frac{c_{t+1}}{c_t}\right)^{-1/\psi}\right]^{\theta} \left[\frac{1}{R_{t+1}^w}\right]^{1-\theta}, \qquad \theta = \frac{1-\gamma}{1-1/\psi}. \tag{10}$$

The parameter θ measures the weight placed on the consumption-growth factor relative to the wealth-return factor. When $\theta = 1$ — which occurs exactly when $\psi = 1/\gamma$, the CRRA case — the wealth return drops out entirely. When $\theta \neq 1$, the wealth return enters the SDF as a separate pricing factor, reflecting the agent's concern for news about future investment opportunities beyond what current consumption growth reveals. The further θ departs from one, the more the wealth return matters as an independent pricing factor.

References

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