Beta Pricing with Frontier Portfolios

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Introduction

In the previous notebook we characterized the minimum-variance frontier (MVF) for an economy with n risky assets. We found that the set of frontier portfolios forms a hyperbola in the (σ, μ) space and that any two frontier portfolios span the entire frontier. In this notebook we exploit the spanning property to establish a fundamental pricing result: the expected return of any asset is determined solely by how much it co-moves with any given frontier portfolio.

The key insight is that every asset can be decomposed into a frontier component and an idiosyncratic residual. The residual, by construction, has zero covariance with all frontier portfolios and therefore carries no compensation in expected returns — it is unpriced risk. What matters for pricing is only the systematic exposure to the frontier.

We develop this idea in three steps. First, we define the idiosyncratic risk of an asset and show it is uncorrelated with every frontier portfolio. Second, we establish that for every frontier portfolio p (other than the minimum variance portfolio) there exists a unique zero-covariance frontier portfolio z. Third, we combine these results to derive the beta-pricing formula

$$\mathsf{E}(r_i) = \mu_z + \beta_i(\mathsf{E}(r_p) - \mu_z),$$

where $\beta_i = \mathsf{Cov}(r_i, r_p)/\sigma^2(r_p)$ and μ_z is the expected return of the zero-covariance portfolio of p. The next notebook shows how this formula simplifies when a risk-free asset is present.[1]

[1] Beta pricing is sometimes presented as a consequence of the law of one price alone. However, our derivation rests on the stronger assumption that asset payoffs are arbitrage-free, as stated in Definition 1 of the previous notebook. No-arbitrage implies the law of one price, but not vice versa; by assuming arbitrage-free payoffs we rule out free-lunch opportunities beyond mere return duplication.

Beta Pricing

Property 1 (Idiosyncratic Risk)**.** *Consider an asset whose expected return is μ_i. We define the idiosyncratic risk of the asset as the difference between its return, and the return of a frontier portfolio that has the same expected return. It turns out that the idiosyncratic risk of any asset is uncorrelated with all frontier portfolios.*

Consider an arbitrary frontier portfolio with expected return μ_p and pick any asset or portfolio with expected return μ_i. The covariance between r_p and r_i is

$$\begin{aligned}
\mathrm{Cov}(r_i, r_p) &= \mathbf{w}_i'\mathbf{V}\mathbf{w}_p \\
&= \mathbf{w}_i'\mathbf{V}(\mathbf{a} + \mu_p\mathbf{b}) \\
&= \mathbf{w}_i'\left[\frac{1}{D}(B\boldsymbol{\iota} - A\mathbf{e}) + \frac{\mu_p}{D}(C\mathbf{e} - A\boldsymbol{\iota})\right] \\
&= \frac{1}{D}\left(B - A\mu_p - A\mu_i + C\mu_i\mu_p\right).
\end{aligned}$$

This shows that two assets with the *same expected return* will have the same covariance with a given frontier portfolio p. This seemingly marginal property is at the heart of modern asset pricing. Given a frontier portfolio, what determines the *expected return* of any asset is how much it covaries with the frontier portfolio regardless of its total risk.

To look at this result in more detail, consider the frontier portfolio $r_{p,i}$ with the same expected return as asset i. Define the residual

$$\varepsilon_i = r_i - r_{p,i}. \tag{1}$$

By construction, the residual has zero mean

$$\mathrm{E}(\varepsilon_i) = \mathrm{E}(r_i) - \mathrm{E}(r_{p,i}) = 0,$$

and if we pick any frontier portfolio p we also have that

$$\mathrm{Cov}(r_p, \varepsilon_i) = \mathrm{Cov}(r_p, r_i) - \mathrm{Cov}(r_p, r_{p,i}) = 0.$$

That is, the residual is unrelated to the returns of frontier portfolios. Since we just saw that the expected return of an asset depends only on how it covaries with a frontier portfolio, the residual is not priced, hence the name idiosyncratic risk.

Property 2 (Zero-Covariance). *For a given frontier portfolio p with expected return μ_p and variance σ_p^2, we can always find (except for the minimum variance portfolio) a unique frontier portfolio z with expected return μ_z that is uncorrelated with p, i.e.*

$$\text{Cov}(r_z, r_p) = 0.$$

Just draw a line that is tangent to the minimum-variance frontier at the point (σ_p, μ_p). The intercept of this line with the vertical axis gives μ_z.

Using the variance expression $\sigma^2 = (B - 2A\mu + C\mu^2)/D$, we can compute the derivative of σ with respect to μ,

$$\begin{aligned} \frac{d\sigma}{d\mu} &= \frac{d\sqrt{\sigma^2}}{d\mu} \\ &= \frac{1}{2\sigma}\frac{d\sigma^2}{d\mu} \\ &= \frac{C\mu - A}{D\sigma}. \end{aligned} \tag{2}$$

Denote by m_p the slope coefficient of the minimum variance frontier at the point (σ_p, μ_p). Equation (2) implies that

$$\frac{\sigma_p}{m_p} = \frac{C\mu_p - A}{D},$$

since $d\sigma/d\mu = 1/(d\mu/d\sigma) = 1/m$.

The equation of the tangent line to the minimum variance frontier at point (σ_p, μ_p) is given by

$$\mu - \mu_p = m_p(\sigma - \sigma_p).$$

Denote by μ_z the intercept of this line with the vertical axis. Then we have that

$$\mu_z = \mu_p - m_p\sigma_p.$$

We can now compute the covariance of r_p and a frontier portfolio r_z with expected return μ_z.

$$\begin{aligned}
\mathrm{Cov}(r_z, r_p) &= \frac{1}{D}\left(B - A\mu_p - A\mu_z + C\mu_z\mu_p\right) \\
&= \frac{1}{D}\left(B - 2A\mu_p + C\mu_p^2\right) - \frac{\mu_p - \mu_z}{D}(C\mu_p - A) \\
&= \sigma_p^2 - m_p\sigma_p\left(\frac{C\mu_p - A}{D}\right) \\
&= \sigma_p^2 - m_p\sigma_p\left(\frac{\sigma_p}{m_p}\right) \\
&= 0.
\end{aligned}$$

Obviously, the covariance of the minimum variance portfolio with any other frontier portfolio is equal to its variance, so it is impossible to find a frontier portfolio that has zero-covariance with it.[2]

Property 3 (Beta Pricing with Frontier Portfolios)**.** *Frontier portfolios contain all the information we need to price assets and carry all the systematic risk of the economy. Just pick any frontier portfolio p with return r_p, and compute its associated zero-covariance portfolio r_z. Then for any asset or portfolio i we have that*

$$r_i = (1 - \beta_i)r_z + \beta_i r_p + \varepsilon_i,$$

where

$$\beta_i = \frac{\mathrm{Cov}(r_i, r_p)}{\sigma^2(r_p)},$$

$$\mathrm{E}(\varepsilon_i) = \mathrm{Cov}(r_p, r_z) = \mathrm{Cov}(r_p, \varepsilon_i) = \mathrm{Cov}(r_z, \varepsilon_i) = 0.$$

Let $\mu_i = \mathrm{E}(r_i)$. We start by re-writing (1) as

$$r_i = r_{p,i} + \varepsilon_i,$$

[2] Geometrically, the tangent to the frontier at the minimum variance portfolio is vertical in (σ, μ) space, so it has no finite intercept with the μ-axis.

and note that $r_{p,i}$ is a frontier portfolio, and hence it can be represented as a portfolio of any two other frontier portfolios. Therefore, just pick an arbitrary frontier portfolio p (but different from the minimum variance portfolio)[3] with expected return μ_p, and find its associated zero-covariance frontier portfolio z. We then form a portfolio composed of both frontier portfolios such that

$$r_{p,i} = (1-\beta_i)r_z + \beta_i r_p,$$

with

$$\beta_i = \frac{\mu_i - \mu_z}{\mu_p - \mu_z}.$$

The choice for β_i guarantees that $\mathrm{E}(r_{p,i}) = \mu_i$. Note that ε_i is uncorrelated with both r_z and r_p, and that r_z and r_p are also uncorrelated. Hence, the covariance of r_i and r_p is given by

$$\begin{aligned}
\mathrm{Cov}(r_i, r_p) &= \mathrm{Cov}((1-\beta_i)r_z + \beta_i r_p + \varepsilon_i, r_p) \\
&= (1-\beta_i)\underbrace{\mathrm{Cov}(r_z, r_p)}_{0} + \beta_i \underbrace{\mathrm{Cov}(r_p, r_p)}_{\sigma^2(r_p)} + \underbrace{\mathrm{Cov}(\varepsilon_i, r_p)}_{0} \\
&= \beta_i \sigma^2(r_p),
\end{aligned}$$

which yields that

$$\beta_i = \frac{\mathrm{Cov}(r_i, r_p)}{\sigma^2(r_p)}.$$

[3]The restriction is necessary because, by Proposition 2, the zero-covariance portfolio z does not exist for the minimum variance portfolio.