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Binomial Pricing

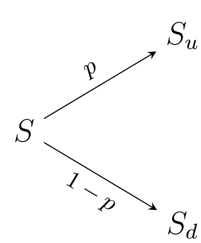
Introduction

Put-call parity says a European call is the same as a European put and a long forward contract on the stock. Without arbitrage opportunities, this relationship is exact and robust to market frictions. It would be nice, however, to derive the price of the European call option *independently* from the price of the European put.

One of the most important results in option pricing is that, under certain conditions, we can replicate the price evolution of the call option from some more basic assets, namely the stock itself and a risk-free bond. To do this, though, we need a stochastic model of the stock price evolution over time.

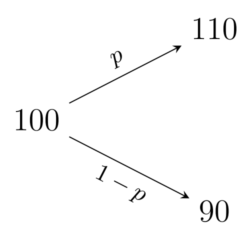
Binomial Trees

One of the easiest ways to describe the evolution of a stock price over time is to use what we call a *binomial tree* in finance. At each point, there are only two possibilities for the future stock price occurring with probability p and $1-p$, respectively. Specifically, if the current stock price is S, then over the next period, the stock price can take the values $S_u > S$ and $S_d < S$.



In the following, we will usually compute $S_u = S \times u$ and $S_d = S \times d$, where u and d are the gross percentage increase and decrease of the stock price over the next period, respectively. We will later relate these quantities to the volatility of stock returns to make the tree consistent with observed data.

Example 1 (One-Period Binomial Tree). The current stock price is \$100. Next period, the asset can go up or down by 10% with probability p and $1 - p$, respectively. In this example $u = 1.10$ and $d = 0.90$.



In the tree, the stock price next period can be either \$110 or \$90. □

The Replicating Portfolio Approach

If the option premium were more expensive than the price of its replicating portfolio, then it would make sense to buy the replicating portfolio, which provides us with a synthetic option, and sell the option for a higher price. If, on the contrary, the cost of the option was lower than its replicating portfolio, it would make sense to buy the option and sell the synthetic option obtained through the replicating portfolio.

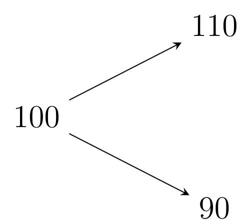
In financial markets, these trading strategies constitute an *arbitrage opportunity*, an easy way to make free money. There are theoretical and empirical arguments about why arbitrage opportunities cannot exist for long.

Theoretically, a strategy that requires zero investment and generates positive payoffs in the future would incentivize anyone who finds it to exploit it without bounds. This behavior would violate any economic equilibrium model of security prices. The demand for the option or the replicating portfolio, whichever is cheapest, would adjust the price quickly until the arbitrage opportunity disappears.

Empirically, when markets function well, arbitrage opportunities do not last long. It is important to note that arbitrage opportunities can last longer in periods of market stress or when trading constraints are significant. Nevertheless, we often observe that derivative prices prevent obvious arbitrage opportunities.

Replicating a Put Option

We will use the replicating portfolio approach to price a European put option to show how the method works. Consider a non-dividend paying stock that currently trades for $100. Over the next six months, the stock can go up or down by 10%.



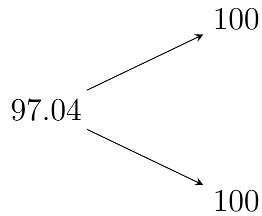
Stock

The interest rate is 6% per year with continuous compounding. What should be the price of a European put option with a maturity of 6 months and a strike price of $100?

We need another financial instrument that pays differently from the stock to replicate the put. Consider then a risk-free bond with a maturity of 6 months and a face value equal to the strike price of the put, i.e., $100. The cost of the bond is:

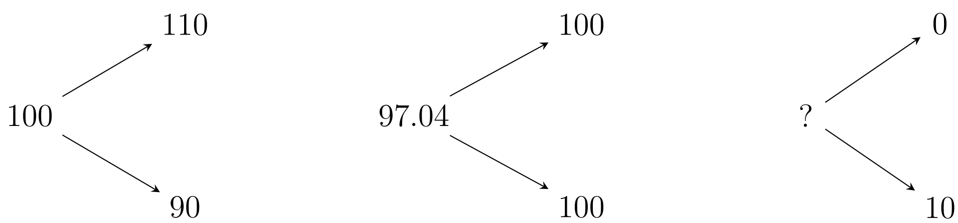
$$B = 100e^{-0.06\times 6/12} = \$97.04.$$

Note that the bond pays $100 at maturity no matter what happens to the stock price in six months. Therefore, the corresponding binomial tree for the bond is:



Bond

The put pays nothing if the stock price is \$110 and pays \$10 if the stock price is \$90. The idea now is to use the stock and the risk-free bond to replicate the payoffs of the put. The figure below depicts the payoffs of the bond and the put for the two different values of the stock.



Stock **Bond** **Put**

Say we purchase N_S units of the stock and N_B units of the bond. Such a portfolio would pay

$$\text{Payoff} = \begin{cases} 110N_S + 100N_B & \text{if } S = 110 \\ 90N_S + 100N_B & \text{if } S = 90 \end{cases}$$

Furthermore, say we choose N_S and N_B such that payoff of the portfolio matches the payoff of the put, i.e.

$$110N_S + 100N_B = 0$$
$$90N_S + 100N_B = 10$$

We can solve for N_S and N_B to find:

$$N_S = \frac{0 - 10}{110 - 90} = -0.50$$
$$N_B = -\frac{110}{100}N_S = (-1.1)(-0.5) = 0.55$$

Therefore, by shorting 0.50 units of the stock and going long 0.55 units of the bond, we can exactly match the payoffs of the put. We usually call the number of shares needed to replicate the option the *delta* (Δ) of the put.

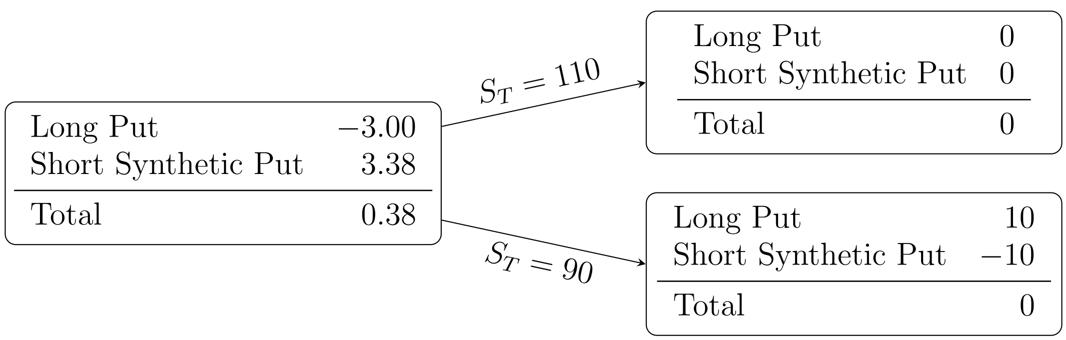
The price of the put must then match the price of the replicating portfolio, otherwise there would be an arbitrage opportunity:

$$P = -0.5 \times 100 + 0.55 \times 97.04 = \$3.38.$$

Example 2. Let's see what would happen in the previous analysis if the put was trading for \$3. Then, it would make sense to buy the traded put and sell a synthetic one, purchasing 0.5 shares of the stock and selling 0.55 units of the risk-free bond.

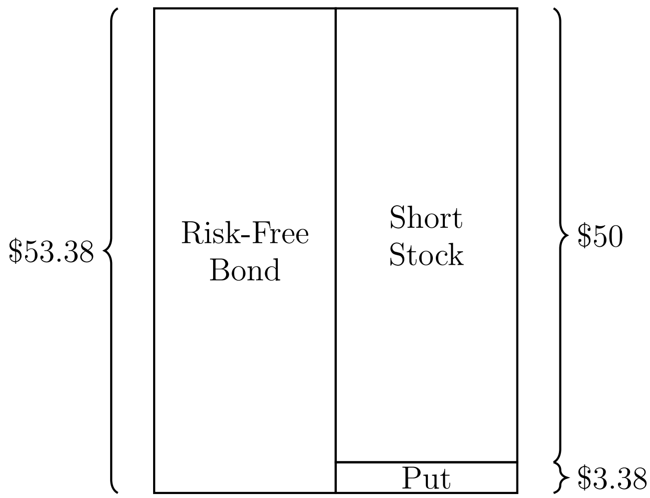
By selling the synthetic put, we collect a cash flow of \$3.38, which is more than enough to buy the put for \$3, leaving a free profit of \$0.38 per transaction. The arbitrage portfolio generates a fully hedged position in six months.

The figure below shows the cash flow of these transactions.



We can see that this would be an obvious arbitrage opportunity since we collect \$0.38 per transaction with no risk afterwards. □

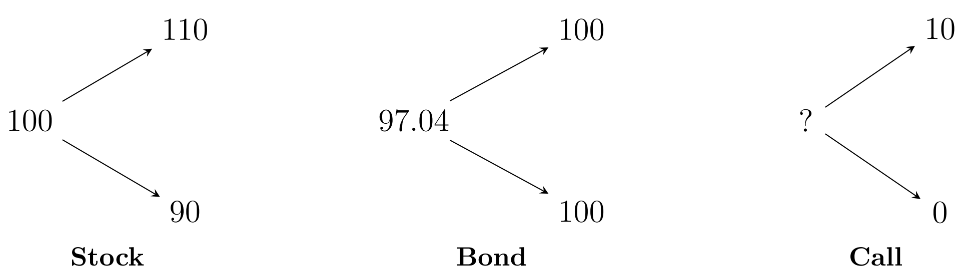
The replication analysis shows that the put is an investment in a risk-free bond financed partly by shorting stocks. Therefore, the put is similar to an equity position of a limited liability entity with a risk-free investment worth \$53.37 financed with a short stock position worth \$50 and \$3.38 of equity.



Since the risk-free position does not depend on the value of the stock, a change in the stock price causes an opposite change in the value of the put. For example, an increase of 1% in the stock reduces the put value by approximately $0.5/3.38 = 14.8\%$. Therefore, the put carries more risk than the underlying stock and correlates inversely with it.

Replicating A Call Option

Consider now a European call option with the same maturity and strike price as the put. The call pays $10 is the stock price is $110 and $0 otherwise.



As before, we replicate the payoffs of the call option by trading the stock and the bond:

$$110N_S + 100N_B = 10$$
$$90N_S + 100N_B = 0$$

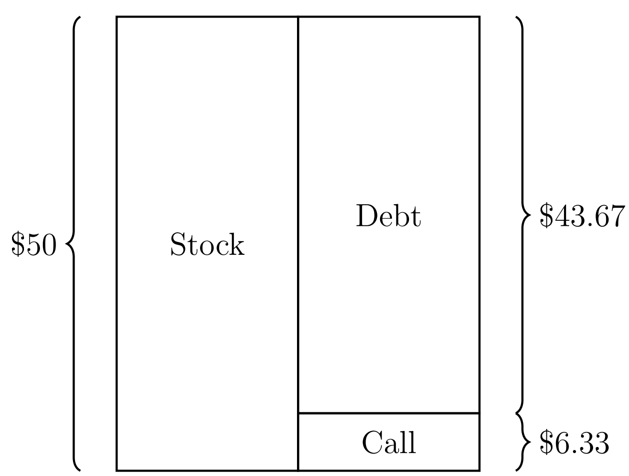
We can solve for N_S and N_B to find:

$$N_S = \frac{10 - 0}{110 - 90} = 0.50$$
$$N_B = -\frac{90}{100} N_S = (-0.90)(0.5) = -0.45$$

Therefore, by buying 0.50 units of the stock and shorting 0.45 units of the bond we can exactly match the payoffs of the call. The price of the call must then match the price of the portfolio, otherwise there would be an arbitrage opportunity:

$$C = 0.5 \times 100 - 0.45 \times 97.04 = \$6.33.$$

The replication analysis reveals that the call can be seen as a levered position on the stock. We could see the call as the equity position of a limited liability entity that has a stock position worth $50 that is financed with $43.67 of debt and $6.33 of equity.

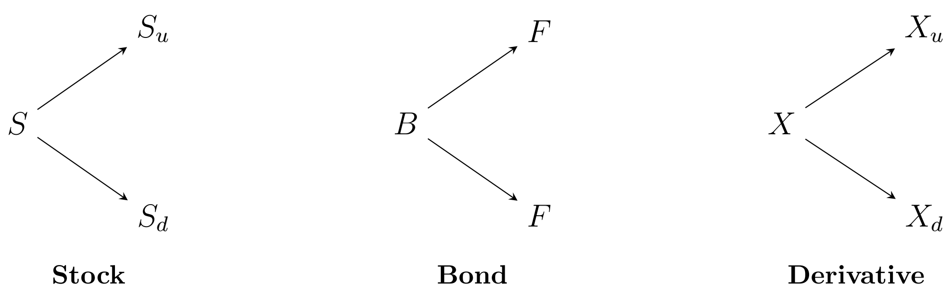


The risk of the call is amplified by the implicit leverage in the position. A 1% decrease in the stock price reduces the value of the call approximately by $0.5/6.33 = 7.9\%$. Thus, the risk of a call option is many times larger than the risk of the underlying asset.

Replicating A Generic Derivative

The analysis so far suggests that we can generalize the replicating approach to price any derivative. By a derivative we mean a security that pays a certain amount X_u is the stock goes up and X_d is the stock price goes down at time T.

We replicate the derivative by trading the stock and a zero-coupon risk-free bond with face-value F and expiring at T as well. If the risk-free rate with continuous compounding is r, the value of the bond today is $B = Fe^{-rT}$.



As before, we start by matching the payoffs of the derivative using N_S shares of the stock and N_B units of the bond:

$$S_u N_S + F N_B = X_u$$

$$S_d N_S + F N_B = X_d$$

Solving for N_S and N_B shows

$$N_S = \frac{X_u - X_d}{S_u - S_d},$$

$$N_B = \frac{X_u - S_u N_S}{F} = \frac{X_d - S_d N_S}{F}.$$

The price of the derivative must then match the price of the portfolio, otherwise there would be an arbitrage opportunity:

$$X = N_S S + N_B B.$$

The number of shares N_S needed to replicate the derivative is called the delta of the instrument. Note that the specific face-value of the risk-free bond is not really relevant since if we choose a different value the number of bonds bought or sold would get adjusted accordingly.

Example 3. A non-dividend paying stock currently trades at $50 and can either increase to $60 or decrease to $40 over the next 6 months. An investment bank is offering to its clients a product that pays in 6 months $1000 if the stock goes up and $200 if the stock goes down. The risk-free rate is 5% per year with continuous compounding. What is the no-arbitrage price of the derivative? How many shares of the stock does the bank needs to buy or sell in order to hedge the derivative?

As we said before, we can choose any face value for the risk-free bond. If we use $100 then the value of the bond is $B = 100e^{-0.05\times 6/12}$. The number of shares needed to hedge the derivative is

$$N_S = \frac{1000 - 200}{60 - 40} = 40,$$

whereas the number of bonds is

$$N_B = \frac{1000 - 60 \times 40}{100} = -14.$$

Finally, the no-arbitrage price of the derivative is

$$40 \times 50 + (-14) \times 100e^{-0.05\times 6/12} = 634.57.$$

Therefore, the bank needs to buy 40 shares of the stock to hedge the derivative and sell it for $634.57. □

The Risk-Neutral Approach

In replicating the payoffs of the option, we never used the actual probabilities. As a matter of fact, these probabilities might even change based on whose thinking about the asset. Since the previous reasoning is silent about the probabilities and the type of investor pricing the asset, we can assume in our reasoning that all investors are risk neutral. Even if this is not true in real markets, such assumption would not affect **the logic** of the replicating-portfolio argument.

The real probabilities are thus irrelevant. In a world populated by risk-neutral investors, all expected payoffs should be discounted at the risk-free rate, regardless of their riskiness. Therefore, the price of any asset X is equal to its expected payoffs discounted at the risk-free rate:

$$X = (X_u q + X_d(1-q))e^{-rT}.$$

In practice, we can compute q by using the price of the stock as:

$$S = (S_u q + S_d(1-q))e^{-rT}$$

which implies that

$$q = \frac{Se^{rT} - S_d}{S_u - S_d}. \tag{1}$$

We can then use the risk-neutral probabilities to compute the price of a call or put option.

An alternative way to think about the risk-neutral probabilities is the following. In a risk-neutral world, investors are indifferent between receiving an expected cash flow $qS_u + (1-q)S_d$ or selling the stock forward for a fixed forward price $F = Se^{rT}$. This means that in a risk-neutral world the forward price is the best forecast of the price of the stock, that is:

$$\underbrace{Se^{rT}}_{\text{Forward Price}} = \underbrace{qS_u + (1-q)S_d}_{\text{Forecast}}. \tag{2}$$

Equation (2) implies that we can also write (1) as

$$q = \frac{F - S_d}{S_u - S_d}$$

where F denotes the forward price of the stock.

Pricing the Call and Put Again

Continuing with our example, the price of the stock should be equal to the expected payoff discounted at the risk-free rate, that is

$$100 = (110q + 90(1-q))e^{-0.06\times 6/12}.$$

We can then reverse-engineer the probability of the stock going up that makes consistent valuations in this world,

$$q = \frac{100e^{0.06\times 6/12} - 90}{110 - 90} = 0.6522.$$

Therefore, the price of the call can be computed as the expected payoff under this risk-neutral probability, discounted at the risk-free rate,

$$C = (10q + 0(1-q))e^{-0.06\times 6/12} = \$6.33.$$

Similarly, for the put we have that:

$$P = (0q + 10(1-q))e^{-0.06\times 6/12} = \$3.38.$$

Of course, the prices are the same as before since both approaches are consistent with each other.

Example 4. A non-dividend paying stock trades at \$50 and over the next 6-months can go up to \$60 or down \$40. The risk-free rate is 6% per year with continuous compounding. Let's compute the price of a European call option expiring in 6 months with strike price \$48.

We start by computing the risk-neutral probability of the stock moving up:

$$q = \frac{50e^{0.06\times 6/12} - 40}{60 - 40}.$$

Therefore, the price of the call is:

$$C = (12q + 0(1-q))e^{-0.06\times 6/12} = 6.71.$$

Note that at this point we could use put-call parity to compute the price of an otherwise equivalent European put. □

Example 5. A non-dividend paying stock trades at \$120 and over the next 3-months can increase or decrease by 10%. The risk-free rate is 5% per year with continuous compounding. What is the price of an asset that pays in 3 months \$100 if the stock increases in price and \$200 otherwise?

We have that the stock can move up to $120 \times 1.10 = 132$ or down to $120 \times 0.90 = 108$.

Therefore, the risk-neutral probability of the stock moving up is:

$$q = \frac{120e^{0.05\times 3/12} - 108}{132 - 108}.$$

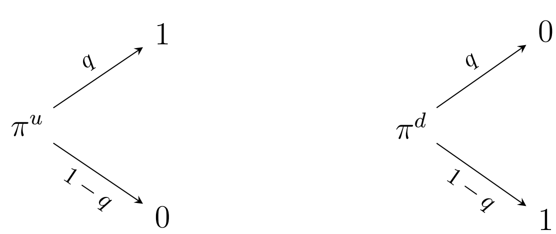
The price of the asset is:

$$X = (100q + 200(1-q))e^{-0.05\times 3/12} = 141.93.$$

The risk-neutral probabilities allow us to compute the price of any derivative or *contingent-claim* written on the stock. □

State Prices

The risk-neutral probabilities are intimately related to the so-called Arrow-Debreu securities depicted below.



The price of each security is then the expected payoff using the risk-neutral probabilities, discounted at the risk-free rate:

$$\pi^u = (1q + 0(1-q))e^{-0.06\times 6/12} = \$0.6329$$
$$\pi^d = (0q + 1(1-q))e^{-0.06\times 6/12} = \$0.3375$$

Example 6. Continuing with Example 5, we have that the price of an asset that pays in 3 months \$1 if the stock price increases and \$0 otherwise is:

$$\pi_u = (1q + 0(1-q))e^{-0.05\times 3/12} = 0.5559,$$

whereas the price of an asset that pays in 3 months \$0 if the stock price increases and \$1 otherwise is:

$$\pi_d = (0q + 1(1-q))e^{-0.05\times 3/12} = 0.4317.$$

Thus, the price of an asset that pays in 3 months \$100 if the stock increases in price and \$200 otherwise is:

$$X = 100\pi_u + 200\pi_d = 141.93,$$

which is the same value as before. □

Practice Problems

Solutions to all problems can be found at lorenzonaranjo.com/fin451.

Problem 1. A stock price is currently \$50. It is known that at the end of 2 months it will be either \$53 or \$48. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 2-month European call option with a strike price of \$49? Use the replicating portfolio argument and indicate the number of shares required to hedge the position.

Problem 2. A stock price is currently \$80. It is known that at the end of 4 months it will be either \$75 or \$85. The risk-free interest rate is 5% per annum with continuous compounding. What is the value of a 4-month European put option with a strike price of

$80? Use the replicating portfolio argument and indicate the number of shares required to hedge the position.

Problem 3. A stock price is currently $40. It is known that at the end of 1 month it will be either $42 or $38. The risk-free interest rate is 8% per annum with continuous compounding. What is the value of a 1-month European call option with a strike price of $39?

Problem 4. A stock price is currently $50. It is known that at the end of 6 months it will be either $45 or $55. The risk-free interest rate is 10% per annum with continuous compounding. What is the value of a 6-month European put option with a strike price of $50?

Problem 5. Consider a non-dividend paying asset that trades for $100. Over the next six months, analysts expect that it could go up to $113 or down to $90. Compute the price of an at-the-money European call option expiring in six months. Assume that the risk-free rate is 7% per year with continuous compounding.

Problem 6. Consider a non-dividend paying asset that trades for $99. Over the next six months, analysts expect that it could go up to $114 or down to $88. Compute the price of an at-the-money European put option expiring in six months. Assume that the risk-free rate is 6% per year with continuous compounding.

Problem 7. The current price of a non-dividend paying stock is $139. Over the next year, it is expected to go up or down by 10% or 12%, respectively. The risk-free rate is 6% per year with continuous compounding. A market-maker of an important investment bank just sold 100 at-the-money European call options (i.e. one contract) expiring in one year to an important client. How many shares of the stock does she need to buy in order to hedge her exposure?

Problem 8. The current price of a non-dividend paying stock is $87. Over the next year, it is expected to go up or down by 11% or 14%, respectively. The risk-free rate is 6% per year with continuous compounding. A market-maker of an important investment bank just sold 100 at-the-money European put options (i.e. one contract) expiring in one year to an important client. How many shares of the stock does she need to sell in order to hedge her exposure?

Problem 9. In the binomial pricing model, the real probabilities of the stock going up or down do not matter to price a derivative written on the stock because:

a. We can replicate the payoffs of the derivative irrespective of the real probabilities.
b. All investors in the real economy are risk-neutral.
c. It gives a reasonable approximation to the no-arbitrage price.
d. We should always assume that the stock can go up or down with equal probability.

Problem 10. In the one-period binomial model, the risk-neutral probability of the stock going up is:

a. Equal to the real probability that the stock should go up.
b. The probability that investors living in a risk-neutral world would use to assess the likelihood of the stock going up.
c. A mathematical artifact that allows us to compute the correct price of a derivative.