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Interest Rates

Introduction

The pricing of financial assets is always relative to some benchmarks that are well-priced. We use risk-free bonds as the relevant benchmarks to compare cash flows occurring at different points in time. Unfortunately, financial markets are rarely fully integrated, so the landscape of interest rates is varied and complex.

Consider a *zero-coupon* bond that pays \$100 in 2 years to start thinking about interest rates. If you were to purchase the bond, you would pay less than \$100. Indeed, most people prefer to consume earlier than later and would like compensation for waiting to receive their money. With this in mind, say that the price of the bond is \$95.

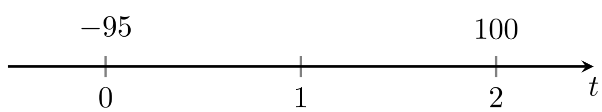

Figure 1: Price and cash flows of a zero-coupon bond with face value \$100 and expiring in 2 years.

Therefore, every dollar paid in two years today costs \$0.95. Thus, a bond that pays for certain \$50,000 in two years today should cost $0.95 \times 50{,}000 = \$47{,}500$.

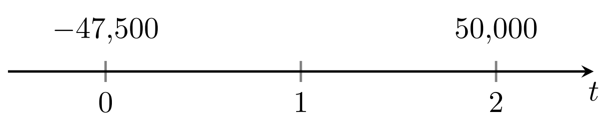

Figure 2: Price and cash flows of a zero-coupon bond with face value \$50,000 and expiring in 2 years.

What if this new bond trades for a different price, say \$47,000? Then, there would be an *arbitrage opportunity*. Note that 500 bonds paying \$100 in two years are equivalent to a single bond paying \$50,000 in two years. Therefore, buying the bond with a face value \$50,000 expiring in two years and short-selling 500 bonds with a face value \$100 expiring on the same date makes sense. This long-short strategy would provide today with a cash flow of \$500 and, in two years, requires us to pay \$50,000 for the bonds that we sold-short, which is precisely offset by the bond we bought.

Selling an asset short involves borrowing the asset first, which requires us to repay it later. Short-selling zero-coupon bonds may be more challenging for a small investor, but it will be easy for a prominent institutional investor such as a hedge fund.

Also, making \$500 in one trade is not a large sum, but the idea here is to lever the transaction as much as possible. If your arbitrage capital is \$100 million, you can short-sell 1,000,000 bonds with face value \$100 and purchase 2,000 bonds with face value \$50,000 expiring in two years. The arbitrage profit gets amplified by 2,000, so you make \$1,000,000 in the trade.

Arbitrage opportunities cannot last long as they generate free money for those exploiting them. The asset sold short should decrease in value, whereas the price of the asset bought should increase. Therefore, without arbitrage opportunities, the bond price with face value \$50,000 must be \$47,500.

We can also express the fact that \$100 paid in two years costs today \$95 as a percentage rate of return per year. Indeed, it must be the case that

$$95(1+r)^2 = 100,$$

if the annualized rate of return is r. Therefore,

$$r = \left(\frac{100}{95}\right)^{1/2} - 1 = 2.60\%.$$

In other words, the implicit *interest rate* paid by the 2-year zero coupon bond is 2.60% per year compounded annually.

As we can see, interest rates are usually derived from bond prices, although some interest rates are set directly in loans. The arbitrage opportunity we analyzed previously allowed us to borrow money at 2.60% and invest it at a higher rate.

In our example, the 2-year bond is free of the risk of default. We say that the 2-year *risk-free rate* is 2.60% per year. Risk-free rates for different maturities are fundamental to price derivatives. Therefore, it is essential to have a good proxy for it. In the next section, I review several types of interest rates found in financial markets and discuss which is the best proxy to use as a risk-free rate in pricing futures, forwards, and options.

Different Types of Interest Rates

Treasury Rates

The United States government issues Bills, Notes, and Bonds through the Department of the Treasury to finance government activities. The demand for Treasury securities is high since, for many investors, Treasury bonds represent an investment free of default risk.

As of June 2023, the total public Treasury debt outstanding is $32.33 trillion. Let's compare this figure with the market capitalization of the U.S. stock market, which is $46.20 trillion as of June 2023. It makes the U.S. Treasury debt market a big contender for worldwide investors.

Notwithstanding the sheer size of the U.S. Treasury market, the demand for Treasury securities often surpasses its supply, which is capped in size by the United States Congress. For this reason, the yield-to-maturity (YTM) of Treasury bonds is usually lower than the rate of a fully collateralized loan. Therefore, Treasury rates are commonly not used as benchmark rates to price derivative securities.

LIBOR

LIBOR rates have been at the heart of the financial system for many decades. The acronym stands for *London Interbank Offered Rate*. For all this time, LIBOR has provided a reference for pricing derivatives, loans, and securities.

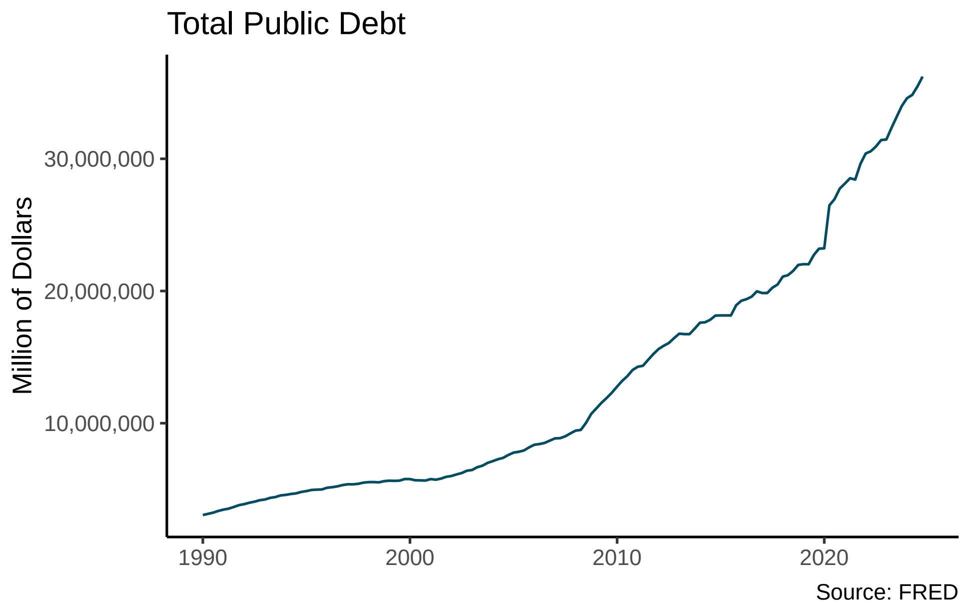

Figure 3: The figure shows the evolution of the Treasury public debt from January 1990 until December 2024.

From the beginning, banks designed LIBOR to proxy for unsecured short-term borrowing between them, with tenors ranging from one day to one year. Also, LIBOR rates existed for different currencies.

Banks used to index large corporate loans to LIBOR. A company that would borrow money from a bank would then reimburse the loan by making quarterly payments indexed to the applicable three-month LIBOR. Many corporate borrowers, however, like to pay a fixed interest instead. Transforming a floating interest rate into a fixed one is possible by combining the loan with an *interest rate swap*. Thus, interest rate swaps were one of the most essential derivatives that used LIBOR as a reference rate.

In response to recommendations and objectives set forth by the Financial Stability Board and the Financial Stability Oversight Council to address risks related to USD LIBOR, the Federal Reserve Board and the New York Fed jointly convened in 2014 the Alternative Reference Rates Committee (ARRC), which is a group of private-market participants whose role is to ensure a successful transition from U.S. dollar (USD) LIBOR to a more robust reference rate. The recommended alternative was the Secured Overnight Financing Rate (SOFR). Duffie and Stein (2015) present a good summary and analysis of the reform.

There were two important reasons to abandon USD LIBOR in favor of SOFR. First, several cases of attempted market manipulation and false reporting of LIBOR came to light in 2012. Second, there has been a decline in the liquidity of interbank unsecured funding post-2009. Therefore, the information from the unsecured financing market might only partially represent the level of risk-free rates.

Even though USD LIBOR is still reported for some tenors, its use has stopped and has been officially replaced by SOFR.

OIS and Overnight Rates

In the United States, the regulation requires banks to maintain cash reserves with the Federal Reserve. When a bank needs to increase its reserve, it usually borrows overnight from another bank that might have a reserve surplus. These brokered transactions' weighted average rate is termed the *effective federal funds rate* (EFFR).

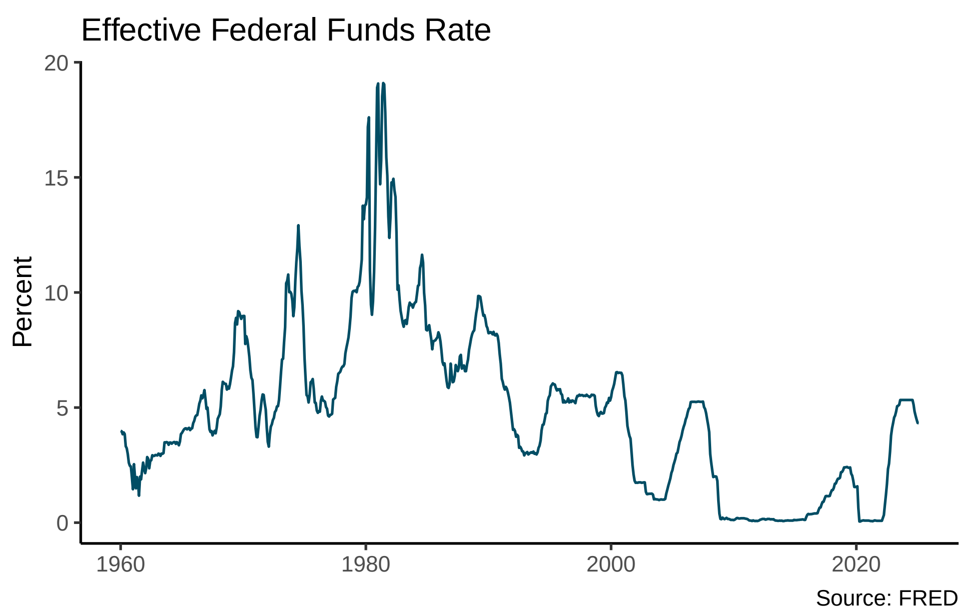

Figure 4: The figure shows the evolution of the effective federal funds rate from January 1960 until December 2024.

We can observe from the figure that there have been periods in which the federal funds rate has been very high, reaching almost 20% per year. From 2009 until 2018, the federal funds rate was nearly zero.

When the Federal Reserve determines the *target federal funds rate*, it implements its policy by ensuring the EFFR is close to its target daily. Since 2009, the Fed has maintained the EFFR between lower and upper limits, as the following figure shows.

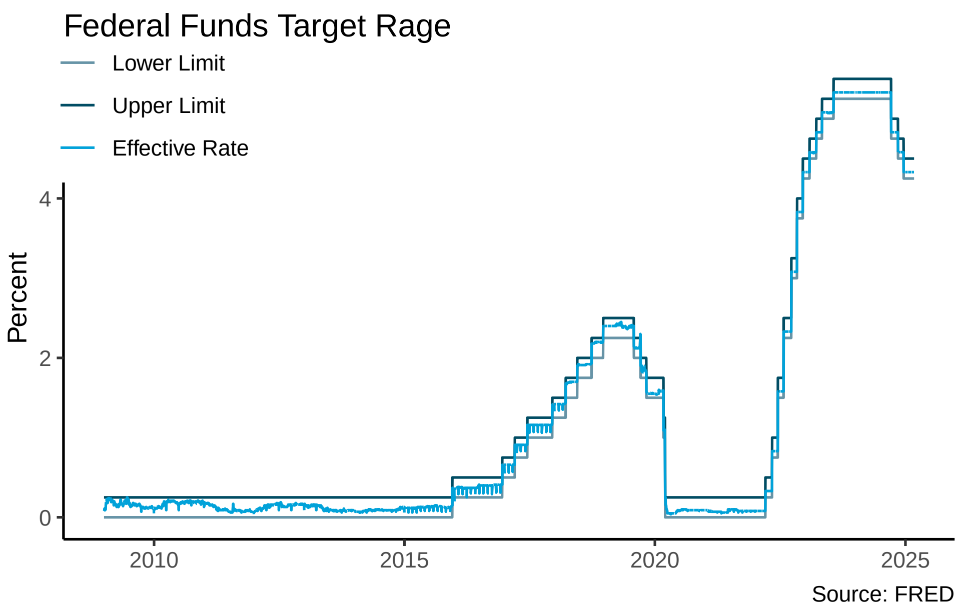

Figure 5: The Federal Reserve makes sure that the EFFR stays within the target range. The figure shows the evolution of the EFFR starting in 2009.

The COVID-19 pandemic has pushed these limits back to where they were after the financial crisis 2008. The recent inflation surge has surprised markets and made the Federal Reserve increase the reference interest rate.

Traditionally, practitioners have used LIBOR and LIBOR-swap rates as proxies for risk-free rates when valuing derivatives. The financial crisis of 2007 put this practice into question. Banks became reluctant to lend to each other, and the TED spread, the difference between the 3-month LIBOR and the 3-month Treasury rate, surged significantly above its average level. It was clear that LIBOR was capturing the credit risk of the underlying banks and, therefore, could not be used to proxy for the risk-free rate.

The alternative chosen by practitioners was the overnight indexed swap rate. An overnight indexed swap (OIS) is an over-the-counter financial contract in which one party pays the compounded EFFR over a certain period, say three months, in exchange for a fixed payment. For years since the financial crisis of 2008, the OIS rate has been a proxy for the risk-free rate (Hull and White 2015). Recently, the ARRC has questioned this practice, citing the OIS market's relative illiquidity. The committee's recommendation was to switch to SOFR.

SOFR and Repo Rates

In a repurchase agreement or repo, a financial institution or trader sells some securities to a counterparty with the agreement to repurchase them back later for a slightly higher price. The implicit interest rate in this transaction is the repo rate. Unlike LIBOR and the EFFR, repo rates are secured borrowing rates.

The weighted average of these repo transactions is called the *secured overnight financing rate* (SOFR). Effective 2022, this rate has replaced LIBOR USD.

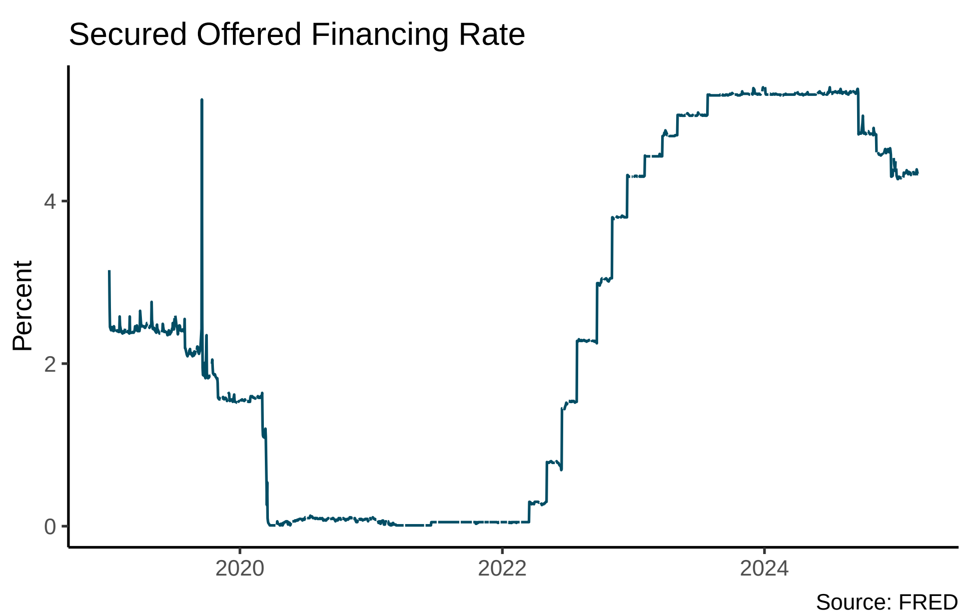

Figure 6: The figure shows the evolution of the secured overnight financing rate (SOFR) from January 2019 until December 2024.

Compounding Frequencies

The notion of *future value* (FV) is fundamental to finance. It allows us to compute how an investment grows over time when you apply a specific rate of interest to it. Unless otherwise stated, we will express all interest rates as a percentage per year. It makes a difference how often you compound the interest rate.

Example 1 (Annual compounding)**.** If the interest rate is 10% per year compounded annually, after a year, $100 invested at this rate will grow to

$$\text{FV} = 100 \times 1.10 = \$110.00.$$

In other words, the future value of $100 after a year is $110 when the interest rate is 10% per year with annual compounding. Consequently, a rate of 10% per year with yearly compounding corresponds to the investment's *effective annual rate* (EAR). □

A prevalent way to express the interest rate is with *annual compounding*. However, practitioners use other ways to express the interest rate in practice. For example, consider the fixed-income market, which in the United States is even more significant than the stock market in terms of market capitalization. Bonds in the United States pay coupons every six months. Consequently, expressing the *yield-to-maturity* (YTM) of Treasury or corporate bonds with *semi-annual compounding* is very common. In this case, we divide the interest rate by two and compound it twice yearly.

Example 2 (Compounding different frequencies)**.** If the interest rate is 10% per year compounded semi-annually, $100 will grow to

$$\text{FV} = 100 \times 1.05^2 = \$110.25.$$

If that interest rate is 10% per year compounded monthly, $100 will grow to

$$\text{FV} = 100 \left(1 + \frac{0.10}{12}\right)^{12} = \$110.47.$$

If the same interest rate is compounded daily, $100 will grow to

$$\text{FV} = 100\left(1 + \frac{0.10}{365}\right)^{365} = \$110.52.$$

You could keep increasing the compounding frequency to see what happens with the future value. □

Example 2 suggests the following rule for the EAR of an investment that uses a specific compounding rule.

Effective Annual Rate

An interest rate r that is compounded n times per year will generate an effective annual rate equal to:

$$\text{EAR} = \left(1 + \frac{r}{n}\right)^{n} - 1$$

The property implies that the EAR increases as we compound more often.

Continuous Compounding

When pricing futures, forwards, options, and other derivatives, it is helpful to use continuous compounding to discount cash flows to keep the same convention when pricing derivatives in continuous time. In this section, we will learn how to use continuously compounded rates to compute present and future values and convert between continuously compounded rates and other conventions used in practice, such as annual or semi-annual compounding.

Indeed, there is a limit to the compounding operation in Example 2:

$$\lim_{n\to\infty} 100\left(1 + \frac{0.10}{n}\right)^{n} = 100e^{0.10} = \$110.52.$$

We call this operation *continuous* compounding, and compounding the interest rate daily is already a pretty good approximation of it. In a spreadsheet, to compute $100e^{0.10}$ you need to type `=100*exp(0.10)`.

More generally, an amount PV invested today in a risk-free deposit account earning an annual interest rate r compounded continuously will grow to $FV = PVe^{rt}$ at time t.

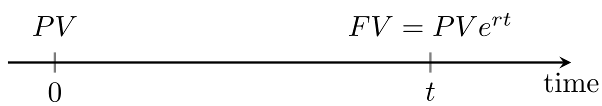

Figure 7: The cashflows of a zero-coupon bond.

We say that FV is the *future value* of PV at time t. Equivalently, we say that $PV = FVe^{-rt}$ is the *present value* of FV.

Continuously-Compounded Interest Rate

If we denote by r the continuously-compounded interest rate, the relationship between present value (PV) and future value (FV) is given by:

$$FV = PVe^{rT} \Leftrightarrow PV = FVe^{-rT}.$$

Example 3 (Computing a future value)**.** If you invest $500 at 6% per year with continuous compounding, in 16 months the balance of your account will be:

$$FV = 500e^{0.06\times16/12} = \$541.64.$$

To perform the previous computation using a spreadsheet software you need to type `=500*exp(0.06*16/12)`.

Example 4 (Computing a present value)**.** You want to know how much you need to have in a bank account today so that in 18 months from now your balance is $2,000. The interest rate is 7% per year with continuous compounding.

The required amount is the present value of \$2,000 for 18 months at 7%:

$$PV = 2000e^{-0.07\times 18/12} = \$1{,}800.65.$$

To perform the previous computation using a spreadsheet software you need to type `=2000*exp(-0.07*18/12)`.

Example 5 (Computing a continuously compounded rate)**.** Say that you invest \$100 in a deposit account and after a year your balance is \$110. What equivalent continuously compounded interest rate would give you the same amount?

If we denote by r the continuously compounded interest rate, it must be the case that:

$$110 = 100e^{r} \Rightarrow r = \ln\left(\frac{110}{100}\right) = 9.53\%.$$

To perform the previous computation using a spreadsheet software you need to type `=ln(110/100)`.

Therefore 10% per year compounded annually is the same as 9.53% per year compounded continuously.

Zero Rates

A zero-coupon bond pays its principal or face-value F at maturity but makes no intermediate payments.

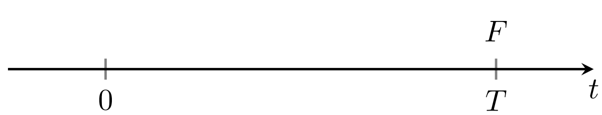

Figure 8: The cashflows of a zero-coupon bond with face value F and expiring at time T.

Even though zero-coupon bonds do not trade actively in financial markets, it is possible to *synthesize* a zero-coupon bond from a portfolio of two coupon bonds.

The price of a zero-coupon bond is determined by discounting its face-value at the relevant interest rate. For a given maturity T, the T-year zero rate, denoted by $r(T)$, is the interest rate that gives the right T-year zero-coupon bond price $Z(T)$. That is, if $r(T)$ is a continuously-compounded rate we must have:

$$Z(T) = Fe^{-r(T)T}.$$

Example 6 (Computing the price of a zero coupon bond)**.** You have the following information for zero rates expressed per year with continuous compounding:

Maturity (months)	1	3	6	9	12
Zero Rate (%)	6.0	6.4	6.6	6.8	7.0

Consider a zero-coupon risk-free bond with face value \$1,000 and expiring in 9 months. To compute the price of the bond, we use the 9-month interest rate which is 6.8%:

$$Z = 1000e^{-0.068\times 9/12} = \$950.28.$$

Example 7 (Computing a zero rate)**.** A 7-year zero-coupon risk-free bond with face value \$1,000 trades for \$650. The 7-year zero rate r expressed with continuous compounding satisfies:

$$\begin{aligned} 1000e^{-r\times 7} &= 650 \\ e^{-r\times 7} &= \frac{650}{1000} \\ -r \times 7 &= \ln\left(\frac{650}{1000}\right) \\ r &= -\frac{1}{7}\ln\left(\frac{650}{1000}\right) \end{aligned}$$

The 7-year zero rate is then 6.15% per year with continuous compounding.

Practice Problems

For solutions go to lorenzonaranjo.com/fin451.

Problem 1. You invest $5,000 at 15% per year with continuous compounding. How much will you have in the account after 14 years?

Problem 2. Consider the following cash flows occurring at the times indicated below:

Time (years)	7	14	21
Cash flow	149	240	184

Compute the present value if the discount rate is 9% per year with continuous compounding.

Problem 3. The interest rate is 8% per year continuously-compounded and assumed to remain constant. Compute the present value of the following cash flows.

Time (years)	2	5	14	19
Cash Flow	100	200	300	150

Problem 4. An investor receives $1,100 in one year in return for an investment of $1,000 now. Calculate the percentage return per year with:

a. Annual compounding
b. Semiannual compounding
c. Monthly compounding
d. Continuous compounding

Problem 5. An effective annual rate (EAR) of 11% per year is equivalent to which rate expressed per year with continuous compounding?

Problem 6. You are considering an investment of $100 today that should grow to $167 in 8 years. What rate of return, expressed per year with continuous compounding, is consistent with this information?

Problem 7. A bank quotes an interest rate of 14% per year with quarterly compounding. Compute an equivalent rate with:

a. Continuous compounding
b. Annual compounding

Problem 8. A deposit account pays 12% per year with continuous compounding, but interest is actually paid quarterly. How much interest will be paid each quarter on a $10,000 deposit?

Problem 9. Suppose that 6-month, 12-month, 18-month, 24-month, and 30-month zero rates are, respectively, 4.0%, 4.2%, 4.4%, 4.6%, and 4.8% per year, with continuous compounding. Estimate the cash price of a bond with a face value of $100 that will mature in 30 months and pays a coupon of 4% per year semiannually.

Problem 10. You have information of cash flows and zero-coupon rates (per year with continuous compounding) for different maturities as shown below:

Time (years)	4	10	17
Zero-coupon rate (%)	5.8	6.9	7.3
Cash flow	387	473	276

Compute the present value of those cash flows.

References

Duffie, Darrell, and Jeremy C. Stein. 2015. "Reforming LIBOR and Other Financial Market Benchmarks." *Journal of Economic Perspectives* 29 (2): 191–212.

Hull, John C., and Alan White. 2015. "OIS Discounting, Interest Rate Derivatives, and the Modeling of Stochastic Interest Rate Spreads." *Journal of Investment Management* 13 (1): 64–83.