Problem Set 2

Instructions: This problem set is due on 12/1 at 11:59 pm Central Time and is an individual assignment. All problems must be handwritten. Write full sentences to answer questions that require an explanation. Scan your work and submit a PDF file.

Problem 1. A non-dividend-paying stock trades for \$60, has a volatility of 30% per year, and has an expected return of $\mu = 12\%$ per year. The risk-free rate is 4% per year with continuous compounding. Consider a European call on the stock with strike \$65 that expires in six months.

a. Use the Black-Scholes model to compute the price of the call.
b. Compute the risk-neutral probability that the call finishes in the money, and the probability under the real distribution of the stock price.
c. The expected payoff of the call under the real distribution is

$$\mathrm{E}\left(\max(S_T - K, 0)\right) = Se^{\mu T}\Phi(\hat{d}_1) - K\Phi(\hat{d}_2),$$

where $\hat{d}_1$ and $\hat{d}_2$ are d_1 and d_2 with μ in place of r. Compute this expected payoff and discount it at the risk-free rate. Is the result above or below the Black-Scholes price? Explain why the expected payoff of the call should not be discounted at the risk-free rate.

Problem 2. A stock index trades at 100 and pays no dividends, and the risk-free rate is 4% per year with continuous compounding. The table below gives Black-Scholes prices of a three-month at-the-money call with strike 100 and a three-month put with strike 85 for several volatilities.

σ	15%	18%	21%	24%	27%	30%	33%
Call ($K = 100$)	3.50	4.09	4.68	5.27	5.87	6.46	7.05
Put ($K = 85$)	0.03	0.09	0.20	0.37	0.59	0.85	1.15

In the market, the call trades for \$4.09 and the put trades for \$0.85.

a. Find the implied volatility of each option.
b. According to the Black-Scholes model, what should the two implied volatilities be? What does the gap tell you about how the market prices the risk of a large fall in the index?
c. A dealer who prices the put with the implied volatility of the call would quote it at $0.09. Who would buy puts from this dealer, and what risk would the dealer end up holding?

Problem 3. A portfolio manager holds $67 million in a fund that tracks the S&P 500 index. The index stands at 6,700, has a dividend yield of 1.3% per year and a volatility of 18% per year, and the risk-free rate is 4% per year with continuous compounding. To protect the portfolio over the next three months, the manager buys a put spread on SPX: long puts with strike 6,500 and short puts with strike 6,000. Each SPX contract is written on 100 times the index.

a. How many contracts of each put does the manager need to cover the portfolio?
b. Compute the price of each put and the total cost of the put spread.
c. What is the largest payoff the spread can deliver, and what is the risk-neutral probability that it pays this maximum?
d. Explain what protection the manager gives up by selling the puts with strike 6,000, and why the manager might accept that.

Problem 4. A German exporter will receive USD 10,000,000 from a US customer in one year. The EUR/USD spot rate is $1.16 per euro, the volatility of the exchange rate is 8% per year, and the interest rates are 3.75% per year in dollars and 2% per year in euros, both continuously compounded.

a. Compute the one-year forward EUR/USD rate.
b. Compute the prices of a one-year EUR call / USD put and a EUR put / USD call, both struck at the forward rate, per euro of notional. Explain why the two prices are equal.
c. The exporter worries that the dollar will weaken. Which of the two options protects the exporter? The exporter buys that option with a strike of $1.20 per euro on the notional that converts USD 10,000,000 at the strike. Compute the notional in euros and the premium, in dollars and in euros at today's spot rate.
d. Show that the price of the call can be written as $C = e^{-rT}[F\Phi(d_1) - K\Phi(d_2)]$, where F is the forward rate and $d_1 = \dfrac{\ln(F/K) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}}$.

Problem 5. A farmer expects to sell 50,000 bushels of corn in three months. The December corn futures trades at $4.50 per bushel with a volatility of 25% per year, and the risk-free rate is 4% per year. To protect the sale, the farmer buys puts on the futures with strike $4.20 and sells calls on the futures with strike $4.90. The options expire in three months and each contract is written on 5,000 bushels. Assume the farmer sells the corn at the futures price on the expiration date of the options, ignoring the basis between the local cash price and the futures.

a. Use Black's model to compute the price of each option and the net cost of the collar for the 50,000 bushels.
b. What is the range of prices per bushel, net of the cost of the collar, that the farmer locks in?
c. The dealer who sold the puts to the farmer and bought the calls hedges with corn futures. Compute the delta of the dealer's position per bushel and the number of futures contracts the dealer buys or sells.

Problem 6. A stock index stands at 6,000 and has a dividend yield of 1.5% per year, and the risk-free rate is 4% per year with continuous compounding. Assume that the index can be bought or sold short at its quoted level as a portfolio of its stocks.

a. Compute the fair price of a futures contract on the index that expires in three months. The futures trades at 6,060. Describe the trades that exploit the gap, and compute the profit per unit of the index at expiration.
b. Three-month European options on the index with strike 6,000 trade for 245 for the call and 190 for the put. Check put-call parity, $C - P = Se^{-\delta T} - Ke^{-rT}$, and describe the trades that exploit the violation. How much do they earn today per unit of the index?
c. A grain merchant proposes to price corn futures in the same way, from the cash price of corn in an elevator carried forward at the risk-free rate plus storage costs. Explain why the argument that works for the index does not give a riskless profit for corn.

Problem 7. A desk sells 10,000 six-month European calls with strike $50 on a non-dividend-paying stock that trades for $50. The volatility is 30% per year and the risk-free rate is 4% per year with continuous compounding. The desk hedges with the replicating portfolio and rebalances once a week.

a. Compute the price of the calls, the number of shares the desk buys, and the amount it borrows. Check that the premium plus the loan pays for the shares.

b. One week later ($\Delta t = 1/52$) the stock trades for \$53. Compute the new value of the calls, the value of the shares minus the loan with interest, and the profit or loss of the desk.
c. Repeat the previous part if the stock trades for \$47 and if it trades for \$50.
d. Explain the pattern in your answers using the shapes of the call price and of the hedge as functions of the stock price.

Normal Distribution Table

The table below computes $\Phi(z) = \mathrm{P}(Z \leq z)$ if $Z \sim \mathcal{N}(0,1)$ and $z \geq 0$. The rows denote the first decimal whereas the columns denote the second decimal. If you need $\Phi(-z)$, you can always use $\Phi(-z) = 1 - \Phi(z)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990