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Class 6

Perfectly Correlated Assets

Class 5 showed that the cross term in the portfolio variance can remove risk, and that how much it removes depends on the weights given the correlation. The extreme case is **perfect correlation**, $\rho_{AB} = \pm 1$, where one asset's return is an exact linear function of the other's. The portfolio variance then becomes a perfect square,

$$\sigma_P^2 = (w_A\sigma_A \pm w_B\sigma_B)^2, \tag{1}$$

with the same sign as the correlation, so weights that set the term in parentheses to zero remove all risk. Using $w_A = 1 - w_B$, the **zero-variance portfolio** is

$$w_B = \begin{cases} \dfrac{\sigma_A}{\sigma_A - \sigma_B} & \text{if } \rho_{AB} = 1, \\ \dfrac{\sigma_A}{\sigma_A + \sigma_B} & \text{if } \rho_{AB} = -1. \end{cases} \tag{2}$$

With $\rho_{AB} = -1$ both weights are positive, and a long position in each asset offsets their fluctuations. With $\rho_{AB} = 1$ one weight is negative: going long the less volatile asset and short the more volatile one, in the right proportion, cancels the risk exactly.

A Synthetic Risk-Free Rate

A portfolio with zero variance earns the same return in every state. If markets offer no arbitrage opportunities, that return must equal the risk-free rate,

$$r_f = w_A\mu_A + w_B\mu_B, \tag{3}$$

with the weights from (2). If the zero-variance portfolio paid more than r_f, you could borrow at r_f and invest in it, earning a riskless profit with no money down. If it paid less, you would short it and lend at r_f. Traders would scale either trade up until prices adjusted, so perfectly correlated assets pin down a **synthetic risk-free rate** implied by their expected returns.

Writing $R_i = r_i - r_f$ for excess returns, (3) says that $w_A R_A + w_B R_B = 0$ in every state, or

$$R_B = \rho_{AB}\frac{\sigma_B}{\sigma_A}R_A. \tag{4}$$

The excess returns of perfectly correlated assets are proportional. Taking expectations, $(\mu_B - r_f)/\sigma_B = \rho_{AB}(\mu_A - r_f)/\sigma_A$. With $\rho_{AB} = 1$ the two assets have the same Sharpe ratio and lie on the same line through r_f. With $\rho_{AB} = -1$ their Sharpe ratios are equal in size and opposite in sign, so they lie on two lines that meet at r_f, one rising and one falling. Figure 1 draws both cases. In each, the zero-variance portfolio is the point where the opportunity set touches the vertical axis.

Building Perfect Correlation

Perfectly correlated pairs are not something you find among traded stocks, but they are easy to build. Combining any risky asset A with the risk-free asset gives a portfolio B with $r_B = r_f + w(r_A - r_f)$, that is,

$$R_B = wR_A, \tag{5}$$

so $\rho_{AB} = 1$ if $w > 0$ and $\rho_{AB} = -1$ if $w < 0$. A levered position in A is perfectly positively correlated with A, and a short position in A with the proceeds invested at r_f is perfectly negatively correlated with it. The set of all such portfolios is the capital allocation line of Class 4 together

with its lower branch: two rays through r_f with slopes equal to plus and minus the Sharpe ratio of A, exactly the shape in Figure 1.

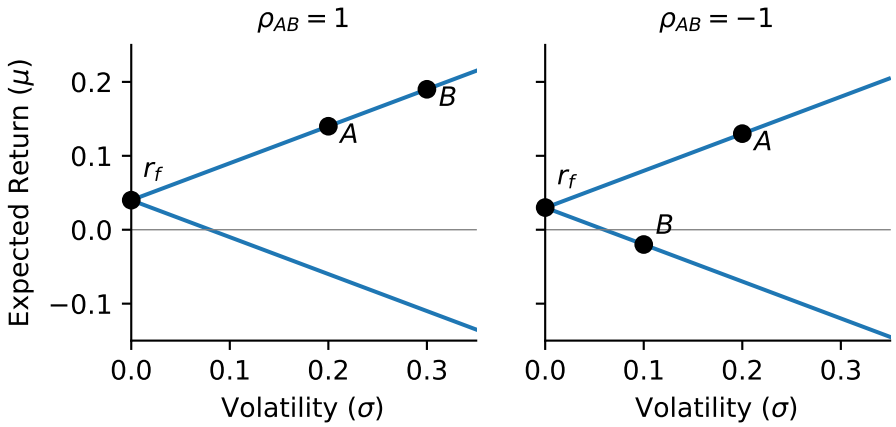


Figure 1: Opportunity sets of two perfectly correlated assets. The zero-variance portfolio earns an implied risk-free rate of 4% on the left and 3% on the right.

Derivatives deliver the same property over short intervals. A call option moves with its underlying asset and a put moves against it, so over a short horizon a call is perfectly positively correlated with the underlying and a put is perfectly negatively correlated with it. Combining the option with the right amount of the underlying creates an overnight risk-free position, which by the argument above must earn r_f. This is the no-arbitrage argument that Black, Scholes, and Merton used in 1973 to price options.

Practice Problems

1. Suppose that there are many assets in the security market and that the characteristics of assets A and B are given as follows:

Asset	Expected Return	Standard Deviation
A	5%	20%
B	20%	30%

The correlation between the asset returns is -1. Suppose that it is possible to invest and borrow at the risk-free rate, r_f. What must be the value of the risk-free rate?