From Beta Pricing to the CAPM

Beta Pricing shows that the expected return of any asset A satisfies

$$\mathrm{E}(r_A) = (1 - \beta_A) r_f + \beta_A \, \mathrm{E}(r_Q)$$

for any mean-variance efficient portfolio Q. This result does not identify which portfolio is efficient, however. The Capital Asset Pricing Model (CAPM) supplies that missing link: in equilibrium, the efficient tangency portfolio Q is the **market portfolio** M.

This identification matters because constructing Q directly requires

$$\boldsymbol{w}_Q \propto \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - r_f \mathbf{1}).$$

In practice, both inputs are difficult to estimate. Historical average returns are noisy estimates of $\boldsymbol{\mu}$, and estimating and inverting a large covariance matrix $\boldsymbol{\Sigma}$ magnifies sampling error. Small changes in either input can therefore produce large changes in the estimated tangency weights.

Market-value weights do not require estimates of expected returns or covariances. The theoretical market portfolio nevertheless contains every risky asset in the investable universe, not just publicly listed stocks, so it is not observed exactly. In applications, investors use broad value-weighted indexes as proxies for M. The CAPM explains why the market portfolio, rather than a statistically estimated tangency portfolio, should be mean-variance efficient.

From Individual Choice to the Market Portfolio

The equilibrium argument rests on a short set of assumptions. Investors evaluate portfolios by their means and variances and share the same expected returns and covariance matrix (**homogeneous expectations**). They can trade the same assets without frictions, borrow or lend at a common risk-free rate, and choose portfolios in markets that clear. These assumptions connect individual portfolio choice to aggregate asset holdings.

Two-Fund Separation

Recall from Optimal Capital Allocation that an investor with risk aversion A who combines the risk-free asset with a risky portfolio Q chooses weight

$$w^* = \frac{\mu_Q - r_f}{A\sigma_Q^2}$$

in Q. The weight w^* depends on risk aversion, but the composition of Q does not. Because investors have homogeneous expectations, they face the same efficient frontier and select the same risky portfolio with the highest Sharpe ratio. More risk-averse investors lend and hold less of Q; less risk-averse investors may borrow and hold more of it. In either case, they hold the risky assets in the same proportions. This result is the **two-fund separation theorem**.

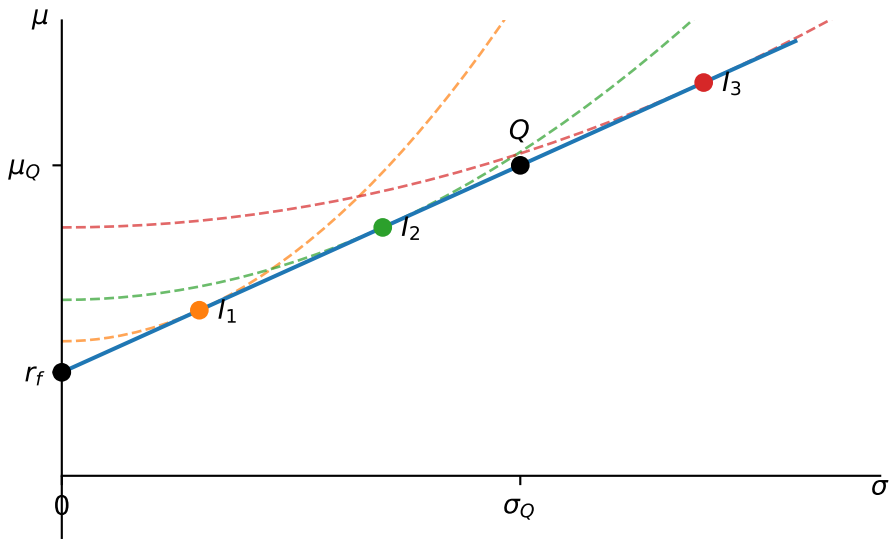


Figure 1: The figure shows three investors with different risk aversion, all holding the tangency portfolio Q combined with the risk-free asset, but in different proportions. Investor 1 lends at the risk-free rate, investor 2 is almost fully invested in Q, and investor 3 borrows to lever up her position in Q. All three hold the risky assets in exactly Q's proportions, and each investor's indifference curve (dashed) is tangent to the CAL at her chosen point, with more risk-averse investors displaying more sharply curved indifference curves.

Aggregation and Market Clearing

Let q_j be asset j's weight in Q, and let W_i be investor i's wealth. Investor i places $w_i^* W_i q_j$ dollars in asset j. Summing across investors gives the aggregate demand for that asset:

$$q_j \sum_i w_i^* W_i.$$

The term multiplying q_j is the same for every risky asset. Consequently, the composition of aggregate risky-asset demand is exactly the composition of Q.

Market clearing requires investors collectively to hold the outstanding supply of every risky asset. If V_j denotes the total market value of asset j, its weight in aggregate risky wealth is

$$\frac{V_j}{\sum_k V_k}.$$

This is also asset j's weight in the **market portfolio** M. Aggregate risky demand has Q's composition and must equal aggregate risky supply, which has M's composition. Therefore,

$$Q = M.$$

This conclusion concerns the *relative weights within the risky portfolio*. It does not require the risk-free asset to be in zero net supply. A positive net supply of borrowing or lending instruments changes the fraction of aggregate wealth invested in risky assets, but not the relative weights of those risky assets.

The assumptions are restrictive. If investors disagree about expected returns or covariances, face different borrowing rates, or cannot trade the same assets, they need not choose the same tangency portfolio. Two-fund separation then fails at the aggregate level, and market clearing no longer implies that M is mean-variance efficient.

The CAPM Pricing Equation

Property 1 (The Capital Asset Pricing Model)**.** *If the assumptions of the CAPM hold, the market portfolio M is mean-variance efficient. The expected excess return of any asset A is therefore proportional to its market beta:*

$$\mathrm{E}(r_A) - r_f = \beta_A\big[\mathrm{E}(r_M) - r_f\big], \tag{1}$$

where

$$\beta_A = \frac{\mathrm{Cov}(r_A, r_M)}{\mathrm{V}(r_M)}$$

measures the asset's exposure to market risk.

Property 1 follows directly from the Beta Pricing proposition after equilibrium identifies Q with M. Equivalently, the CAPM expected-return equation can be written as

$$\mathrm{E}(r_A) = r_f + \beta_A\big[\mathrm{E}(r_M) - r_f\big] = (1 - \beta_A)r_f + \beta_A\,\mathrm{E}(r_M).$$

The market risk premium $\mathrm{E}(r_M) - r_f$ is the expected return investors require for bearing one unit of market risk. An asset with $\beta_A = 1$ has the same expected excess return as the market, whereas an asset with $\beta_A < 1$ has a smaller expected excess return. Idiosyncratic volatility does not enter the equation because it can be diversified away.

Example 1 (Computing a Stock Beta)**.** Suppose the correlation between stock A and the market is 0.6. If the annual standard deviation of A's return is 40% and the annual standard deviation of the market return is 20%, then

$$\begin{aligned}\beta_A &= \frac{\mathrm{Cov}(r_A, r_M)}{\mathrm{V}(r_M)} = \frac{\sigma_A \sigma_M \rho_{A,M}}{\sigma_M^2} \\ &= \frac{\sigma_A \rho_{A,M}}{\sigma_M} = \frac{0.40 \times 0.60}{0.20} = 1.2.\end{aligned}$$

A one-percentage-point change in the market return is associated with a 1.2-percentage-point change in A's return through its systematic component. □

Example 2 (Computing an Expected Return)**.** If the risk-free rate is 5%, $\beta_{DELL} = 1.3$, and $\mathrm{E}(r_M) = 14\%$, then the CAPM predicts

$$\mathrm{E}(r_{DELL}) = 0.05 + 1.3(0.14 - 0.05) = 16.7\%.$$

The 16.7% is Dell's expected annual **total return**, including both price changes and distributions such as dividends. It is an average required return, not a guarantee of the return that will be realized next year. □

Systematic and Firm-Specific Risk

Under the CAPM, an asset's return can be decomposed into a systematic component linked to the market and a firm-specific, or idiosyncratic, component:

$$r_A - r_f = \beta_A(r_M - r_f) + \varepsilon_A, \tag{2}$$

where $\mathrm{E}(\varepsilon_A) = 0$ and $\mathrm{Cov}(\varepsilon_A, r_M) = 0$. The beta term captures systematic risk, which remains in a diversified portfolio. The residual ε_A captures firm-specific risk, which diversification can reduce.

This equation also explains how beta is estimated. A regression of an asset's excess returns on the market's excess returns has slope coefficient

$$\beta_A = \frac{\mathrm{Cov}(r_A, r_M)}{\mathrm{V}(r_M)}.$$

The regression residual estimates ε_A. In empirical work, the intercept is allowed to differ from zero so that researchers can test the CAPM rather than impose it.

Because the systematic and residual components are uncorrelated, the asset's total variance is

$$\sigma_A^2 = \beta_A^2\sigma_M^2 + \sigma^2(\varepsilon_A). \tag{3}$$

The first term is systematic variance; the second is firm-specific variance. Only the exposure measured by β_A, not the amount of firm-specific variance, determines expected return under

the CAPM.

Example 3 (Computing the Residual Risk)**.** Suppose stock A has an annual return standard deviation of 40%, a market beta of 1.1, and the market has an annual return standard deviation of 20%. Its residual variance is

$$\begin{aligned}\sigma^2(\varepsilon_A) &= \sigma_A^2 - \beta_A^2\sigma_M^2 \\ &= 0.40^2 - 1.1^2(0.20^2) = 0.1116.\end{aligned}$$

Therefore, the annual standard deviation of the firm-specific return is

$$\sigma(\varepsilon_A) = \sqrt{0.1116} = 33.41\%.$$

□

Interpreting Empirical Evidence

The CAPM is a statement about expected returns, which cannot be observed directly. Realized returns vary around their expectations, so a stock can earn more or less than the CAPM prediction over a particular month or year without violating the model. Empirical tests instead use many realized returns to estimate whether average returns are related to beta as the CAPM predicts.

A persistent deviation can have several interpretations. The asset may be mispriced, the market portfolio may not be mean-variance efficient because the CAPM's assumptions fail, or the value-weighted index used by the researcher may be an imperfect proxy for the theoretical market portfolio. This market-proxy problem is central to Roll (1977): because the theoretical market portfolio cannot be observed, a test based on an index also tests whether that index is an adequate proxy for M.

This ambiguity is the **joint-hypothesis problem**: every empirical test of market efficiency is also a test of the asset-pricing model used to define the correct expected return. Rejecting the CAPM relation therefore rejects the combined hypothesis, not necessarily market efficiency alone.

Practice Problems

These problems give you a chance to practice the concepts introduced in this chapter. Try to solve each one on your own before expanding the solution.

Solutions to all problems can be found at https://lorenzonaranjo.com/class-materials/investments/.

Problem 1 (Two Investors, One Risky Portfolio)**.** Nora Sato and Theo Lindgren are clients of the same advisor and share her forecasts. The tangency portfolio Q holds 50% in Alder Technologies (ALD), 30% in Birch Energy (BRE), and 20% in Cobalt Health (CBH). It has expected return $\mu_Q = 11\%$ and volatility $\sigma_Q = 20\%$, and T-bills yield 3%. Nora has $200,000 and risk aversion $A = 2$; Theo has $500,000 and risk aversion $A = 4$.

a. Compute each client's optimal weight w^* in Q.
b. How many dollars does each client invest in ALD, BRE, CBH, and T-bills?
c. Compare the two clients' holdings of the three stocks. What does the comparison illustrate?
d. Suppose Nora and Theo were the only investors, and ALD, BRE, and CBH the only risky assets. What would the weights of the market portfolio be?

Problem 2 (Market Weights and Market Clearing)**.** A small stock market lists three companies: Harlow Bank (HRB), with a market capitalization of $60 billion; Ives Motors (IVM), worth $30 billion; and Juno Media (JNM), worth $10 billion. Investors satisfy the assumptions of the CAPM.

a. What are the weights of the tangency portfolio?
b. Rafael Mendes has $1,000,000 and invests 80% of it in the tangency portfolio. How much does he hold in each stock?
c. The next day, Ives Motors' share price rises 20% and the other prices do not change. Compute the new market weights. Does Rafael need to trade to keep holding the market portfolio?

Problem 3 (Finding Mispriced Stocks on the Security Market Line)**.** Isabel Moreau, an analyst at Corran Partners, uses a risk-free rate of 4% and an expected market return of 10%. Her forecasts for three stocks are:

Stock	Beta	Forecast Expected Return
Kestrel Airlines (KSA)	1.4	13%
Linden Foods (LDF)	0.7	7%
Mercer Steel (MCS)	1.0	10%

a. Compute the expected return the CAPM requires for each stock, and the difference between Isabel's forecast and that requirement.
b. Which stocks plot above the security market line and which below? Which would Isabel consider underpriced?

Problem 4 (Beta and Residual Risk from a Regression)**.** A regression of Ostrander Media's (OSM) monthly excess returns on the market's excess returns gives an R^2 of 0.36 and a positive slope. OSM's annual volatility is 40%, the market's is 20%, the risk-free rate is 4%, and the market risk premium is 6%.

a. Using the fact that R^2 is the fraction of OSM's variance explained by the market, compute OSM's beta.
b. Compute OSM's firm-specific volatility $\sigma(\varepsilon_{OSM})$.
c. What expected return does the CAPM assign to OSM? Does its firm-specific risk affect the answer?

Problem 5 (The Cost of Capital for a New Project)**.** Thalassa Petroleum (THP) has a beta of 1.4. It is considering a wind farm that costs $10 million today and is expected to pay $11 million in one year. Wind farms with long-term power contracts have a beta of about 0.8. The risk-free rate is 4% and the market risk premium is 6%.

a. Compute the project's NPV using the CAPM discount rate that matches the project's risk.
b. A manager proposes discounting the project at THP's own cost of equity instead. What NPV does that give, and which decision is right?

Problem 6 (Pricing a Stock with the CAPM)**.** Analysts expect Penrose Instruments (PNI) to pay a dividend of $2 next year and to trade at $60 right after paying it. PNI's beta is 1.25, the risk-free rate is 3%, and the market risk premium is 7%.

a. What price should PNI trade at today?
b. A change in PNI's customer base raises its beta to 1.5, with the same expected cash flows. What happens to the price?

Problem 7 (Diversifying Away Firm-Specific Risk)**.** Wren Ellison builds an equally weighted portfolio of 20 stocks. Each stock has a beta of 1 and a firm-specific volatility of 30%, and the residuals are uncorrelated across stocks. The market's volatility is 18%, the risk-free rate is 4%, and the market risk premium is 6%.

a. Compute the volatility of a single stock and of the portfolio.
b. Compute the expected return of a single stock and of the portfolio. Why does diversification lower risk without lowering expected return?

Problem 8 (Evaluating a Fund Manager)**.** Over the past ten years, the Pryor Capital Opportunity Fund earned an average annual return of 14% with a beta of 1.3 against the S&P 500. Over the same period, T-bills averaged 3% and the S&P 500 averaged 10%.

a. What average return does the CAPM predict for the fund? Compute the fund's alpha.
b. The fund's marketing materials claim the alpha proves the manager is skilled. Give three reasons the evidence might not support that claim.

Problem 9 (The Equilibrium Market Risk Premium)**.** Suppose all investors in an economy have the same risk aversion $A = 3$, and borrowing and lending happen only between investors, so the risk-free asset is in zero net supply. The market portfolio has a volatility of 18%.

a. Market clearing requires the average investor to hold exactly 100% of wealth in the market portfolio. Use the optimal weight w^* to find the market risk premium consistent with equilibrium.
b. During a financial crisis, market volatility rises to 22% while risk aversion stays the same. What happens to the market risk premium, and what must happen to stock prices today for it to rise?

Roll, Richard. 1977. "A Critique of the Asset Pricing Theory's Tests Part I: On Past and Potential Testability of the Theory." *Journal of Financial Economics* 4 (2): 129–76.