Formula Sheet

1. If a return r equals r_s in state s with probability p_s, its expected return and variance are

$$\mathrm{E}(r) = \sum_s p_s r_s \quad \text{and} \quad \mathrm{V}(r) = \sum_s p_s \left(r_s - \mathrm{E}(r)\right)^2$$

   For two returns r_A and r_B, the covariance is

$$\mathrm{Cov}(r_A, r_B) = \sum_s p_s \left(r_{A,s} - \mathrm{E}(r_A)\right)\left(r_{B,s} - \mathrm{E}(r_B)\right)$$

2. For a portfolio between two securities, the return, expected return and variance are

$$\begin{aligned} r &= w_A r_A + w_B r_B \\ \mathrm{E}(r) &= w_A\,\mathrm{E}(r_A) + w_B\,\mathrm{E}(r_B) \\ \mathrm{V}(r) &= w_A^2\,\mathrm{V}(r_A) + w_B^2\,\mathrm{V}(r_B) + 2w_A w_B\,\mathrm{Cov}(r_A, r_B) \end{aligned}$$

3. The covariance between the returns of two securities can be expressed as

$$\mathrm{Cov}(r_A, r_B) = \sigma_{AB} = \sigma_A \sigma_B \rho_{AB}$$

4. Given two risky assets A and B, the weights of the minimum variance portfolio are

$$w_A = \frac{\sigma_B^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}} \quad \text{and} \quad w_B = \frac{\sigma_A^2 - \sigma_{AB}}{\sigma_A^2 + \sigma_B^2 - 2\sigma_{AB}}$$

5. Given a risk-free asset and a risky asset with expected return μ_Q and variance σ_Q^2, a portfolio that invests a fraction w of wealth in the risky asset has

$$\mu = r_f + w(\mu_Q - r_f) \quad \text{and} \quad \sigma = w\sigma_Q$$

The optimal fraction of wealth to invest in the risky asset for an investor with utility function $U = \mu - \frac{1}{2}A\sigma^2$ is

$$w^* = \frac{\mu_Q - r_f}{A\sigma_Q^2}$$

6. In the CAPM, the expected return of a security is

$$\mathrm{E}(r) = r_f + \beta(\mathrm{E}(r_M) - r_f) \quad \text{where} \quad \beta = \frac{\mathrm{Cov}(r, r_M)}{\mathrm{V}(r_M)} = \frac{\rho_{rM}\,\sigma(r)}{\sigma_M}$$

7. The Sharpe ratio is given by

$$\text{Sharpe ratio} = \frac{\mathrm{E}(r) - r_f}{\sigma(r)}$$

8. In the index model, the excess returns $R_i = r_i - r_f$ of security i are modeled as

$$R_i = \alpha_i + \beta_i R_M + e_i \quad \text{where} \quad \sigma_i^2 = \beta_i^2\sigma_M^2 + \sigma^2(e_i)$$

The alpha and the R-square of the regression are

$$\alpha_i = \mathrm{E}(R_i) - \beta_i\,\mathrm{E}(R_M) \quad \text{and} \quad R^2 = \frac{\beta_i^2\sigma_M^2}{\sigma_i^2}$$

For two distinct securities $i \neq j$, the covariance of returns is

$$\mathrm{Cov}(R_i, R_j) = \beta_i\beta_j\sigma_M^2$$

A portfolio with weights w_i has

$$\alpha_P = \sum_i w_i\alpha_i, \quad \beta_P = \sum_i w_i\beta_i \quad \text{and} \quad \sigma^2(e_P) = \sum_i w_i^2\sigma^2(e_i)$$