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Class 1

Probability Spaces and Random Variables

A discrete **probability space** (Ω, P) is a finite or countably infinite set of outcomes Ω together with probabilities $\mathrm{P}(\omega) \geq 0$ that add up to one. An **event** is a subset $A \subset \Omega$, and its probability is

$$\mathrm{P}(A) = \sum_{\omega \in A} \mathrm{P}(\omega).$$

Unions, intersections and complements of events translate "or", "and" and "not". If A and B are disjoint, $\mathrm{P}(A \cup B) = \mathrm{P}(A) + \mathrm{P}(B)$, and in particular $\mathrm{P}(A^C) = 1 - \mathrm{P}(A)$.

A **random variable** X assigns a real number $X(\omega)$ to each outcome. Its **expectation** and **variance** are

$$\mathrm{E}(X) = \sum_{\omega \in \Omega} X(\omega)\mathrm{P}(\omega), \qquad \mathrm{V}(X) = \mathrm{E}\left((X - \mathrm{E}\,(X))^2\right) = \mathrm{E}(X^2) - \mathrm{E}(X)^2,$$

and the **covariance** and **correlation** of X and Y are

$$\mathrm{Cov}(X, Y) = \mathrm{E}(XY) - \mathrm{E}(X)\mathrm{E}(Y), \qquad \rho_{X,Y} = \frac{\mathrm{Cov}(X, Y)}{\sigma_X \sigma_Y} \in [-1, 1].$$

Linear Combinations

For $Z = \alpha X + \beta Y$, expectations are linear while variances pick up the covariance:

$$\mathrm{E}(Z) = \alpha\ \mathrm{E}(X) + \beta\ \mathrm{E}(Y), \qquad \mathrm{V}(Z) = \alpha^2\ \mathrm{V}(X) + \beta^2\ \mathrm{V}(Y) + 2\alpha\beta\ \mathrm{Cov}(X, Y). \qquad (1)$$

If $X_1, \ldots, X_n$ are uncorrelated with common variance σ^2, the sum has variance $n\sigma^2$ and the average has variance σ^2/n. A sum of n shocks therefore drifts a distance of order $\sqrt{n}$, the scaling that will turn a random walk into Brownian motion.

Joint and Marginal Distributions

The **probability mass function** (pmf) of X is $p_X(x) = \mathrm{P}(X = x)$, so that $\mathrm{E}(X) = \sum_x x\, p_X(x)$. The **joint pmf** $p_{X,Y}(x, y) = \mathrm{P}(X = x, Y = y)$ contains everything about X and Y. Summing over one variable recovers the other's pmf,

$$p_X(x) = \sum_y p_{X,Y}(x, y), \qquad p_Y(y) = \sum_x p_{X,Y}(x, y),$$

and expectations of functions of both variables are $\mathrm{E}(f(X, Y)) = \sum_x \sum_y f(x, y)\, p_{X,Y}(x, y)$.

Conditioning and Independence

The **conditional probability** of A given B, for $\mathrm{P}(B) > 0$, is

$$\mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}.$$

Learning B discards the outcomes outside B and rescales the rest. If $B_1, B_2, \ldots$ partition Ω, the **law of total probability** reads

$$\mathrm{P}(A) = \sum_k \mathrm{P}(A \mid B_k)\mathrm{P}(B_k). \tag{2}$$

Events A and B are **independent** if $\mathrm{P}(A \cap B) = \mathrm{P}(A)\mathrm{P}(B)$, and random variables are independent if their joint pmf factors, $p_{X,Y}(x, y) = p_X(x)\, p_Y(y)$. Then $\mathrm{E}(f(X)g(Y)) = \mathrm{E}(f(X))\mathrm{E}(g(Y))$ for any functions f and g, so independent variables are uncorrelated. The converse fails: X uniform on $\{-1, 0, 1\}$ and $Y = X^2$ are uncorrelated, yet Y is a function of X.

Information and Conditional Expectation

A random variable splits Ω into the cells on which it takes each value. Observing it reveals which cell contains the true outcome and nothing more. The events that are unions of cells form the **information set** it generates. A random variable is **known** given an information set $\mathcal{F}$ when it is constant on every cell.

If $\mathcal{F}$ is generated by the partition $C_1, C_2, \ldots$, the **conditional expectation** of Y given $\mathcal{F}$ averages Y over the cell that occurred,

$$\mathrm{E}(Y \mid \mathcal{F}) = \frac{\mathrm{E}(Y \mathbf{1}_{C_k})}{\mathrm{P}(C_k)} \qquad \text{on } C_k. \tag{3}$$

It is a random variable known given $\mathcal{F}$: our best forecast of Y with the information in $\mathcal{F}$. Four properties do almost all the work in this course.

1. **Linearity.** $\mathrm{E}(aY + bW \mid \mathcal{F}) = a\,\mathrm{E}(Y \mid \mathcal{F}) + b\,\mathrm{E}(W \mid \mathcal{F})$.
2. **Taking out what is known.** If W is known given $\mathcal{F}$, then $\mathrm{E}(WY \mid \mathcal{F}) = W\,\mathrm{E}(Y \mid \mathcal{F})$.
3. **Independence.** If Y is independent of $\mathcal{F}$, then $\mathrm{E}(Y \mid \mathcal{F}) = \mathrm{E}(Y)$.
4. **Tower property.** If $\mathcal{G} \subset \mathcal{F}$, then $\mathrm{E}(\mathrm{E}(Y \mid \mathcal{F}) \mid \mathcal{G}) = \mathrm{E}(Y \mid \mathcal{G})$, and in particular $\mathrm{E}(\mathrm{E}(Y \mid \mathcal{F})) = \mathrm{E}(Y)$.

Filtrations

A **filtration** $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 \subset \cdots$ describes information that accumulates and is never forgotten. With independent coin flips $X_n = \pm 1$ and $\mathcal{F}_n$ generated by the first n flips, each flip splits every cell in two, as in Figure 1. The random walk $S_n = X_1 + \cdots + X_n$ is known at time n and the next flip is independent of $\mathcal{F}_n$, so

$$\mathrm{E}(S_{n+1} \mid \mathcal{F}_n) = S_n + \mathrm{E}(X_{n+1}) = S_n. \tag{4}$$

A process whose best forecast is its current value is a **martingale**.

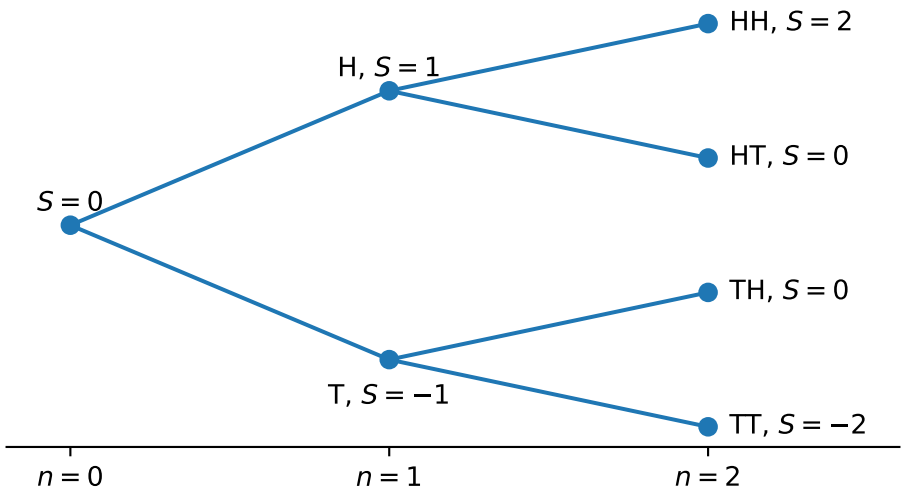


Figure 1: Two flips of a fair coin. Each node is a cell of the partition generating $\mathcal{F}_n$ and shows the value of the random walk S_n.

Practice Problems

1. A portfolio holds two corporate bonds that default independently over the next year with probabilities 0.10 and 0.20. Let N be the number of defaults.

   a. Find the probability mass function of N.
   b. Compute $\mathrm{E}(N)$ and $\mathrm{V}(N)$.
   c. Given that at least one bond defaults, what is the probability that both default?

2. The economy expands next year with probability 0.7 and enters a recession with probability 0.3. A firm defaults with probability 2% in an expansion and 10% in a recession.

   a. Use the law of total probability to compute the probability that the firm defaults.
   b. Given that the firm defaults, what is the probability that the economy was in a recession?

3. Two random variables satisfy $\mathrm{V}(X) = 4$, $\mathrm{V}(Y) = 9$ and $\mathrm{V}(X + Y) = 7$.

   a. Compute $\mathrm{Cov}(X, Y)$ and $\rho_{X,Y}$.
   b. Compute $\mathrm{V}(X - Y)$ and $\mathrm{V}(2X - Y)$.

4. The table below describes two random variables on a sample space with four outcomes.

Outcome	Probability	X	Y
ω_1	0.20	4	1
ω_2	0.30	2	1
ω_3	0.10	6	0
ω_4	0.40	0	0

   a. Write down the partition generated by Y and list the events in $\mathcal{F}_Y$.
   b. Is Y^2 known given $\mathcal{F}_Y$? Is XY? Explain.
   c. Compute $\mathrm{E}(X \mid \mathcal{F}_Y)$ on each cell and verify that $\mathrm{E}(\mathrm{E}(X \mid \mathcal{F}_Y)) = \mathrm{E}(X)$.
   d. Use the properties of conditional expectation to compute $\mathrm{E}(XY \mid \mathcal{F}_Y)$.
   e. Are X and Y independent? Explain.