Class 5

Martingales

A process $\{M_n\}$ is **adapted** to a filtration $\{\mathcal{F}_n\}$ if M_n is known given $\mathcal{F}_n$. It is a **martingale** if, in addition, for all n,

$$\mathrm{E}(M_{n+1} \mid \mathcal{F}_n) = M_n. \tag{1}$$

Given everything known at time n, the best forecast of the next value is the current value: the process does not drift up or down. By the tower property, a martingale also has $\mathrm{E}(M_n) = \mathrm{E}(M_0)$ for every n.

Three Families of Martingales

Let $X_1, X_2, \ldots$ be independent and let $\mathcal{F}_n$ be generated by $X_1, \ldots, X_n$. The variables need not be identically distributed. Each family below follows from taking out what is known and replacing $\mathrm{E}(\cdot \mid \mathcal{F}_n)$ by $\mathrm{E}(\cdot)$ for functions of X_{n+1} alone.

1. **Sums.** If $\mathrm{E}(X_n) = 0$, then $S_n = S_0 + \sum_{i=1}^{n} X_i$ is a martingale.
2. **Squares.** If, in addition, $\mathrm{V}(X_n) = \sigma^2$, then $M_n = S_n^2 - n\sigma^2$ is a martingale. Expanding $(S_n + X_{n+1})^2$, the cross term has conditional mean zero and X_{n+1}^2 adds σ^2 on average, which the term $-n\sigma^2$ removes.
3. **Products.** If $X_n > 0$ and $\mathrm{E}(X_n) = 1$, then $M_n = X_1 X_2 \cdots X_n$, with $M_0 = 1$, is a martingale, since $\mathrm{E}(M_n X_{n+1} \mid \mathcal{F}_n) = M_n \, \mathrm{E}(X_{n+1})$.

Betting Strategies

Let $M_k - M_{k-1}$ be the gain from a one-dollar bet on round k, and let the gambler stake H_k dollars on that round. The stake is chosen before the round is played, so H_k is known given $\mathcal{F}_{k-1}$: the strategy is **predictable**. Starting from W_0, wealth after n rounds is the **martingale transform**

$$W_n = W_0 + \sum_{k=1}^{n} H_k(M_k - M_{k-1}). \tag{2}$$

In finance, M_k is a (discounted) price, H_k is the number of shares held from $k-1$ to k, and W_n is the value of a self-financing strategy.

> **Betting Strategies Cannot Beat a Martingale**
>
> If $\{M_n\}$ is a martingale and $\{H_k\}$ is predictable with each H_k bounded, then $\{W_n\}$ is a martingale and $\mathrm{E}(W_n) = W_0$ for every n.

The proof is one line: $\mathrm{E}(W_{n+1} \mid \mathcal{F}_n) = W_n + H_{n+1}(\mathrm{E}\,(M_{n+1} \mid \mathcal{F}_n) - M_n) = W_n$. In continuous time, these sums become the Itô integral.

The **doubling strategy** bets one dollar on a fair coin, doubles the stake after every loss, and stops at the first win. If the first win comes on round k, the gambler has lost $2^{k-1} - 1$ dollars and then wins 2^{k-1}, a net gain of one dollar, as in Figure 1. A win comes with probability one, yet over any n rounds $\mathrm{E}(W_n) = (1 - 2^{-n}) - 2^{-n}(2^n - 1) = 0$. The sure profit is paid for by an unbounded line of credit.

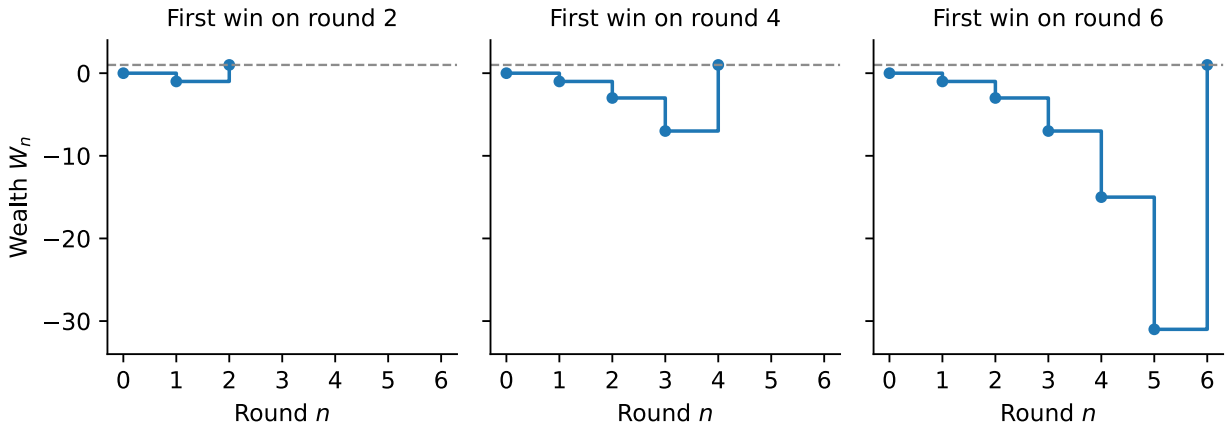


Figure 1: Wealth of the doubling strategy when the first win comes on round 2, 4 or 6. Every path ends one dollar ahead, but the losses before the win double with each round.

Stopping Times and Stopped Martingales

A random time τ with values in $\{0, 1, 2, \ldots\} \cup \{\infty\}$ is a **stopping time** if $\{\tau \le n\} \in \mathcal{F}_n$ for every n: whether we have stopped by time n is known at time n. The truncated time $n \wedge \tau = \min\{n, \tau\}$

is always finite, and the **stopped process** $M_{n\wedge\tau}$ freezes M at time τ. Stopping is itself a betting strategy, one dollar per round while the game runs and nothing afterwards:

$$M_{n\wedge\tau} = M_0 + \sum_{k=1}^{n} \mathbf{1}_{\{\tau\geq k\}}(M_k - M_{k-1}). \tag{3}$$

The stake $\mathbf{1}_{\{\tau\geq k\}}$ is predictable because $\{\tau \geq k\}$ is the complement of $\{\tau \leq k-1\} \in \mathcal{F}_{k-1}$, so a stopped martingale is a martingale and $\mathrm{E}(M_{n\wedge\tau}) = \mathrm{E}(M_0)$ for every n.

Optional Stopping

If τ is finite, $M_{n\wedge\tau} \to M_\tau$ as $n \to \infty$, but the expectations need not converge: the doubling strategy has $W_\tau = 1$ and $W_0 = 0$.

Optional Stopping Theorem

Let $\{M_n\}$ be a martingale and τ a stopping time with $\mathrm{P}(\tau < \infty) = 1$. If there is a random variable Y with $\mathrm{E}(Y) < \infty$ such that $|M_{n\wedge\tau}| \leq Y$ for all n, then

$$\mathrm{E}(M_\tau) = \mathrm{E}(M_0). \tag{4}$$

In particular, the result holds when the stopped process is bounded by a constant.

The doubling strategy fails the condition: the wealth just before the win is $-(2^{\tau-1} - 1)$, and $\mathrm{E}(2^{\tau-1}) = \sum_k 2^{k-1}2^{-k} = \infty$.

Ruin Probabilities Again

For the fair random walk with $S_0 = 0$ and $\tau = \min\{n > 0 : S_n = A \text{ or } S_n = -B\}$, the stopped walk is bounded by $\max(A, B)$, so optional stopping gives $\mathrm{E}(S_\tau) = 0$, that is, $A\,\mathrm{P}(S_\tau = A) - B(1 - \mathrm{P}(S_\tau = A)) = 0$ and $\mathrm{P}(S_\tau = A) = B/(A + B)$. The martingale $S_n^2 - n$ is dominated by $\max(A^2, B^2) + \tau$, which is integrable because $\mathrm{E}(\tau) < \infty$, so $\mathrm{E}(S_\tau^2) = \mathrm{E}(\tau)$, which gives $\mathrm{E}(\tau) = AB$.

With a biased coin, S_n is not a martingale, but each factor $(q/p)^{X_i}$ is positive with mean $p\frac{q}{p} + q\frac{p}{q} = 1$, so the product $(q/p)^{S_n}$ is a martingale, bounded between $(q/p)^{-B}$ and $(q/p)^{A}$ when stopped. Optional stopping, together with the martingale $S_n - n(p - q)$, gives

$$P_A = \mathrm{P}(S_\tau = A) = \frac{(q/p)^B - 1}{(q/p)^{A+B} - 1}, \qquad \mathrm{E}(\tau) = \frac{AP_A - B(1 - P_A)}{p - q}. \tag{5}$$

The same pattern returns with Poisson processes and Brownian motion: an exponential martingale delivers the ruin probability.

Practice Problems

1. Let $X_1, X_2, \ldots$ be independent fair coin flips, equal to 1 or -1 with probability $1/2$, let $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, and let $\mathcal{F}_n$ be generated by $X_1, \ldots, X_n$.

   a. Compute $\mathrm{E}(S_{n+1}^2 \mid \mathcal{F}_n)$. Is $\{S_n^2\}$ a martingale?
   b. Show that $\{S_n^3 - 3nS_n\}$ is a martingale.
   c. Find the constant c such that $\{c^n 2^{S_n}\}$ is a martingale.

2. A stock trades at a price $M_0 = 100$ that is a martingale. Each period the price rises or falls by \$10 with probability $1/2$, for two periods. A trader buys one share at time 0. If the price falls in the first period, the trader "buys the dip" and holds two shares over the second period; if it rises, the trader sells and holds nothing over the second period. The trader starts with $W_0 = 0$ and borrows at zero interest to buy.

   a. Write down H_1 and H_2 and explain why the strategy is predictable.
   b. Compute W_2 on each of the four price paths, the probability that the trader makes money, and $\mathrm{E}(W_2)$.
   c. Compute $\mathrm{E}(W_2 \mid \mathcal{F}_1)$ after a fall in the first period, and compare it with W_1.

3. A market maker's inventory changes each period by $+1$, 0 or -1 with probability $1/3$ each, independently, starting from zero. The market maker hedges as soon as the inventory reaches $+4$ or -2. Assume τ is finite with $\mathrm{E}(\tau) < \infty$.

   a. Show that the inventory S_n is a martingale and find σ^2 such that $S_n^2 - n\sigma^2$ is a martingale.
   b. Use optional stopping to compute the probability that the inventory reaches $+4$ first.
   c. Use optional stopping to compute the expected number of periods until the market maker hedges. Explain why it differs from $AB = 8$.

4. A trader's strategy gains \$1 with probability $p = 0.6$ and loses \$1 with probability $q = 0.4$ on each trade, independently. The trader stops after making \$3 or after losing \$3.

   a. Between which two values does the stopped martingale $(q/p)^{S_{n\wedge\tau}}$ lie? Why does this matter for optional stopping?
   b. Use optional stopping to compute the probability that the trader makes \$3 first.
   c. Use the martingale $S_n - n(p - q)$ to compute the expected number of trades.