Class 7

From Random Walks to Brownian Motion

Prices move all the time by small amounts, which the Poisson process cannot capture. Start from fair coin flips $X_i = \pm 1$, divide time into slots of length Δt, and move up or down by $\sqrt{\Delta t}$ at the end of each slot. At time $t = m\Delta t$ the **scaled random walk** is $B_t^{(n)} = \sqrt{\Delta t}\,(X_1 + \cdots + X_m)$, with $n = 1/\Delta t$ steps per unit of time, and

$$\mathrm{E}(B_t^{(n)}) = 0, \qquad \mathrm{V}(B_t^{(n)}) = t, \qquad [B^{(n)}, B^{(n)}]_t = \sum_{j=1}^{m} (\sqrt{\Delta t}\, X_j)^2 = t.$$

The step $\sqrt{\Delta t}$ is the only one that keeps the variance at t: smaller steps make the walk collapse and larger ones make it explode. The characteristic function $\cos(u\sqrt{\Delta t})^{t/\Delta t}$ converges to $e^{-\frac{1}{2}u^2 t}$, so $B_t^{(n)}$ converges to $\mathcal{N}(0, t)$, and increments over disjoint intervals stay independent in the limit.

Brownian Motion

A **Brownian motion** $\{B_t\}$ satisfies $B_0 = 0$, has independent increments, has $B_t - B_s \sim \mathcal{N}(0, t - s)$ for $s < t$, and has continuous paths with probability one. The first two properties are those of the Poisson process, the normal increment replaces the Poisson one, and continuity rules out jumps. Writing $B_t = B_s + (B_t - B_s)$ for $s \le t$ gives

$$\mathrm{E}(B_t) = 0, \qquad \mathrm{V}(B_t) = t, \qquad \mathrm{Cov}(B_s, B_t) = \min(s, t). \tag{1}$$

The correlation between B_s and B_t is $\sqrt{s/t}$, which depends on the ratio of the dates rather than on the gap between them. Over a short interval the increment $\mathrm{d}B_t$ has standard deviation $\sqrt{\mathrm{d}t}$, far larger than $\mathrm{d}t$, so a Brownian path is never flat.

Martingales of Brownian Motion

With $\mathcal{F}_t$ generated by the path up to t, each future increment $\Delta B = B_t - B_s$ is independent of $\mathcal{F}_s$, with $\mathrm{E}(\Delta B) = 0$, $\mathrm{E}(\Delta B^2) = t - s$ and $\mathrm{E}(e^{\theta \Delta B}) = e^{\frac{1}{2}\theta^2 (t-s)}$. These three facts show that

$$B_t, \qquad B_t^2 - t, \qquad Z_t = \exp\left(\theta B_t - \frac{1}{2}\theta^2 t\right) \tag{2}$$

are martingales, with proofs identical line by line to those for the Poisson process. The exponential martingale has the form $e^{\theta B_t - tK(\theta)}$, where $K(\theta) = \frac{1}{2}\theta^2$ is the CGF of B_1.

Hitting a Barrier

Default, knock-out options and drawdown limits all ask whether a process reaches a level. Model a log value as a **Brownian motion with drift** $X_t = \mu t + \sigma B_t$, and let τ be the first time it reaches $-a < 0$ or $b > 0$. Applying (2) with $\theta\sigma$ in place of θ and choosing $\theta = -\gamma$, where $\gamma = 2\mu/\sigma^2$, removes the time term and makes $e^{-\gamma X_t}$ a martingale. Stopping it at τ, where X_τ equals $-a$ or b exactly because paths are continuous, gives

$$\mathrm{P}(X_\tau = -a) = \frac{1 - e^{-\gamma b}}{e^{\gamma a} - e^{-\gamma b}}. \tag{3}$$

If $\mu > 0$, letting $b \to \infty$ gives the probability $e^{-\gamma a}$ of ever reaching $-a$, and as $\mu \to 0$ the formula tends to $b/(a+b)$, the fair gambler's ruin probability. For the insurer the surplus jumps past zero, so Lundberg's argument gives only a bound. Here continuity makes the answer exact. Figure 1 shows paths stopped at the two barriers.

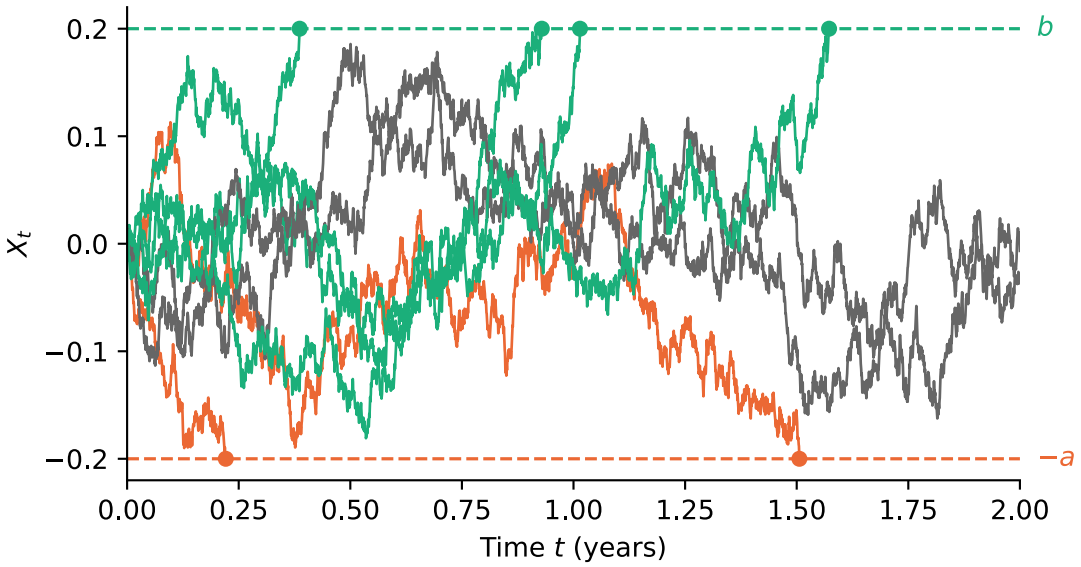


Figure 1: Eight simulated paths of $X_t = \mu t + \sigma B_t$ with $\mu = 0.05$ and $\sigma = 0.20$, each stopped when it reaches $-a = -0.2$ or $b = 0.2$. By (3) a path reaches $-a$ first with probability 0.38.

Quadratic Variation

For a partition $0 = t_0 < \cdots < t_n = T$ with mesh $\|\Pi\|$, the total variation adds up $|f(t_{j+1}) - f(t_j)|$ and the quadratic variation adds up $(f(t_{j+1}) - f(t_j))^2$. A smooth function moves by an amount proportional to $t_{j+1} - t_j$, so it has finite total variation and zero quadratic variation. Brownian motion moves by about $\sqrt{t_{j+1} - t_j}$, and the sum of squared increments Q_Π satisfies

$$\mathrm{E}(Q_\Pi) = T, \qquad \mathrm{V}(Q_\Pi) \le 2\|\Pi\|T, \qquad [B, B]_T = T. \tag{4}$$

The limit is essentially the same on every path, unlike the random $[N, N]_t = N_t$ of the Poisson process. Together with sums of $\mathrm{d}B\,\mathrm{d}t$ and $(\mathrm{d}t)^2$ that vanish, this gives the **multiplication table**

$$(\mathrm{d}B)^2 = \mathrm{d}t, \qquad (\mathrm{d}B)(\mathrm{d}t) = 0, \qquad (\mathrm{d}t)^2 = 0.$$

Since $Q_\Pi \le M_\Pi \sum_j |B_{t_{j+1}} - B_{t_j}|$, where M_Π is the largest move and tends to zero, the **total variation of Brownian motion is infinite** and its paths are smooth on no interval. This is why sums $\sum f(t_j)(B_{t_{j+1}} - B_{t_j})$ that approximate an integral against B depend on where in each subinterval the integrand is evaluated, and why stochastic calculus needs its own rules.

Practice Problems

1. A model of intraday trading moves the log price of a stock up or down by h every period, with probability $1/2$ each, and there are $n = 10{,}000$ periods per year.

   a. Find the step size h that gives the log price a variance of $\sigma^2 = 0.04$ over one year.
   b. Compute the quadratic variation of the log price over one year. Does it depend on the path?
   c. Use the normal limit to approximate the probability that the log price rises by more than 0.3 over the year.
   d. Compute the total distance the log price travels over the year, the sum of the absolute values of all its moves, and compare it with the typical size of the change in the log price over the year. What happens to this distance as $n \to \infty$ with σ^2 fixed?

2. Let $\{B_t\}$ be a Brownian motion, and suppose that $B_1 = 0.5$ has been observed.

   a. Use (2) to compute $\mathrm{E}(B_3 \mid \mathcal{F}_1)$, $\mathrm{E}(B_3^2 \mid \mathcal{F}_1)$, and $\mathrm{E}(e^{B_3} \mid \mathcal{F}_1)$.
   b. Find the constant c such that $e^{2B_t - ct}$ is a martingale, and use it to compute $\mathrm{E}(e^{2B_t})$.
   c. Decide whether each process is a martingale, and justify your answer: (i) tB_t, (ii) $B_t^2 - t + 5$, (iii) $e^{B_t - t}$.

3. A trader's cumulative log return on a position follows $X_t = \mu t + \sigma B_t$, with $\mu = 0.10$ and $\sigma = 0.20$ per year. The trader closes the position with a stop-loss when the log return reaches -0.10 and takes profits when it reaches 0.20.

   a. Compute γ, and the probability that the stop-loss is triggered before the profit target.
   b. Repeat the calculation for a trader with no skill, $\mu = 0$, and comment on how much the drift matters.
   c. Suppose the trader drops the profit target. Compute the probability that the stop-loss is ever triggered.
   d. Without a profit target, how far below zero must the stop-loss be set for the probability of ever triggering it to be 10%?