Class 8

Trading Gains and the Itô Integral

Throughout, $\{B_t\}$ is a Brownian motion and $\mathcal{F}_t$ is the information generated by its path up to time t. A process $\{b_t\}$ is **adapted** if b_t is known given $\mathcal{F}_t$.

An investor who holds b_{t_j} units of an asset whose price is a Brownian motion from t_j to t_{j+1}, choosing each position before seeing the next price change, earns

$$I_t = \sum_{j=0}^{n-1} b_{t_j}\left(B_{t_{j+1}} - B_{t_j}\right). \tag{1}$$

The holding is evaluated at the *left* endpoint of each period. For a general adapted integrand, the **Itô integral** is the mean-square limit of these sums as the partition gets finer,

$$\int_0^t b_s \, \mathrm{d}B_s = \lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} b_{t_j}\left(B_{t_{j+1}} - B_{t_j}\right). \tag{2}$$

Because Brownian paths have infinite total variation, the evaluation point matters. Summing $B_{t_j}\Delta B_j = \frac{1}{2}(B_{t_{j+1}}^2 - B_{t_j}^2) - \frac{1}{2}(\Delta B_j)^2$, the first term telescopes and the second converges to half the quadratic variation, so

$$\int_0^T B_s \, \mathrm{d}B_s = \frac{1}{2}\left(B_T^2 - T\right). \tag{3}$$

The right endpoint would give $\frac{1}{2}(B_T^2 + T)$, the gains of an investor who picks each position after seeing the price move.

Properties of the Itô Integral

Each increment ΔB_j has mean zero and is independent of $\mathcal{F}_{t_j}$, while b_{t_j} is known at t_j. From this single fact, if $\mathrm{E}\left(\int_0^t b_s^2 \, \mathrm{d}s\right) < \infty$, then $I_t = \int_0^t b_s \, \mathrm{d}B_s$ satisfies:

1. **Martingale.** $\{I_t\}$ is a martingale with continuous paths, so $\mathrm{E}(I_t) = 0$.
2. **Itô isometry.** $\mathrm{E}\left(I_t^2\right) = \mathrm{E}\left(\int_0^t b_s^2 \,\mathrm{d}s\right)$.
3. **Quadratic variation.** $[I, I]_t = \int_0^t b_s^2 \,\mathrm{d}s$.

A position of b_s units adds $b_s^2 \,\mathrm{d}s$ to the variance of the gains, and no strategy based on past information turns a fair game into a favorable one. When the integrand is a deterministic function $f(s)$, each approximating sum is a sum of independent normals, so

$$\int_0^t f(s) \,\mathrm{d}B_s \sim \mathcal{N}\left(0, \int_0^t f(s)^2 \,\mathrm{d}s\right). \tag{4}$$

If we only require $\int_0^t b_s^2 \,\mathrm{d}s < \infty$ path by path, the integral still has zero drift but may fail to be a martingale: it is a **local martingale**.

Itô Processes

An **Itô process** combines a time integral and an Itô integral,

$$X_t = X_0 + \int_0^t a_s \,\mathrm{d}s + \int_0^t b_s \,\mathrm{d}B_s, \qquad \mathrm{d}X_t = a_t \,\mathrm{d}t + b_t \,\mathrm{d}B_t, \tag{5}$$

with adapted **drift** a_t and **volatility** b_t. Over a short interval, $\mathrm{E}(\,\mathrm{d}X_t \mid \mathcal{F}_t) = a_t \,\mathrm{d}t$ and $\mathrm{V}(\,\mathrm{d}X_t \mid \mathcal{F}_t) = b_t^2 \,\mathrm{d}t$. Drift and volatility are determined by the paths, so an Itô process is a martingale only if its drift is zero. With the multiplication table $(\,\mathrm{d}t)^2 = 0$, $(\,\mathrm{d}t)(\,\mathrm{d}B_t) = 0$ and $(\,\mathrm{d}B_t)^2 = \mathrm{d}t$,

$$(\,\mathrm{d}X_t)^2 = b_t^2 \,\mathrm{d}t, \qquad [X, X]_t = \int_0^t b_s^2 \,\mathrm{d}s.$$

The drift moves X by amounts of order $\mathrm{d}t$ and leaves no trace in the quadratic variation.

Itô's Formula

If $f(x, t)$ is twice continuously differentiable in x and once in t, then $Y_t = f(X_t, t)$ is an Itô process with

$$\mathrm{d}Y_t = \frac{\partial f}{\partial t} \,\mathrm{d}t + \frac{\partial f}{\partial x} \,\mathrm{d}X_t + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (\,\mathrm{d}X_t)^2. \tag{6}$$

Substituting $\mathrm{d}X_t$ and $(\mathrm{d}X_t)^2 = b_t^2\,\mathrm{d}t$,

$$\mathrm{d}Y_t = \left(\frac{\partial f}{\partial t} + a_t\frac{\partial f}{\partial x} + \frac{1}{2}b_t^2\frac{\partial^2 f}{\partial x^2}\right)\mathrm{d}t + b_t\frac{\partial f}{\partial x}\,\mathrm{d}B_t. \tag{7}$$

The term $\frac{1}{2}b_t^2\frac{\partial^2 f}{\partial x^2}\,\mathrm{d}t$ is the **Itô correction**. In practice, expand f to second order in $\mathrm{d}X$ and first order in $\mathrm{d}t$, then simplify with the multiplication table. Three cases with $X_t = B_t$:

- $\mathrm{d}(B_t^2) = 2B_t\,\mathrm{d}B_t + \mathrm{d}t$, which recovers (3) in one line. The drift of B_t^2 is one, so $B_t^2 - t$ is a martingale, as Figure 1 shows.
- $Z_t = \exp\left(\theta B_t - \frac{1}{2}\theta^2 t\right)$ satisfies $\mathrm{d}Z_t = \theta Z_t\,\mathrm{d}B_t$: the Itô correction exactly offsets $-\frac{1}{2}\theta^2 t$, so Z_t is a martingale.
- $\mathrm{d}(tB_t) = B_t\,\mathrm{d}t + t\,\mathrm{d}B_t$ has no correction, and gives $\int_0^T B_s\,\mathrm{d}s = \int_0^T (T-s)\,\mathrm{d}B_s \sim \mathcal{N}(0, T^3/3)$.

Since Itô integrals have mean zero, taking expectations of the integral form of Itô's formula leaves only the drift: $\mathrm{E}(Y_t) = Y_0 + \mathrm{E}\left(\int_0^t \text{drift}_s\,\mathrm{d}s\right)$.

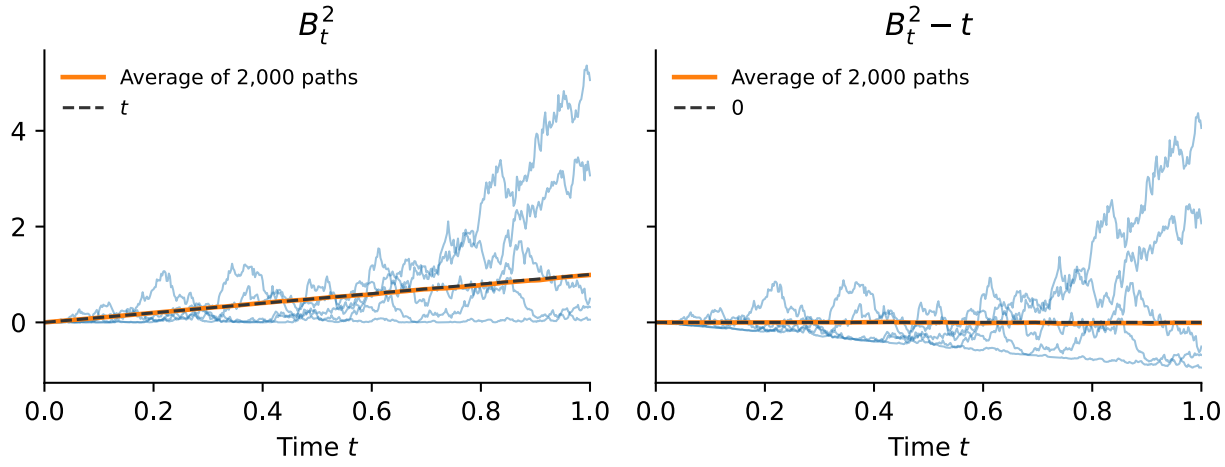


Figure 1: Five simulated paths of B_t^2 and of $B_t^2 - t$ on $[0, 1]$, with the average over 2,000 paths. Itô's formula gives B_t^2 a drift of one, so its average grows like t, while subtracting t removes the drift and leaves a martingale whose average stays at zero.

Practice Problems

1. The price of an asset follows a Brownian motion B_t, measured in dollars. A trader holds one unit from time 0 to time 1. At time 1 the trader looks at the price and, for the period from 1 to 2, holds two units if $B_1 > 0$ and nothing otherwise. Let I_2 be the trading gains at time 2.

   a. Write I_2 as a sum of the form (1) and explain why the strategy is adapted.
   b. Compute $\mathrm{E}(I_2)$ and $\mathrm{E}(I_2 \mid \mathcal{F}_1)$.
   c. Use the Itô isometry to compute $\mathrm{V}(I_2)$.
   d. A second trader holds one unit from 0 to 1 and, for the period from 1 to 2, holds two units if $B_2 > B_1$ and nothing otherwise. Compute the expected gains of this trader and explain why this strategy is not allowed.

2. A fund manager holds two units of an asset whose price follows a Brownian motion during the first half of the year and cuts the position to one unit for the second half. The gains over the year are $I = \int_0^1 f(s)\,\mathrm{d}B_s$, with $f(s) = 2$ for $s < \frac{1}{2}$ and $f(s) = 1$ for $s \geq \frac{1}{2}$.

   a. Find the distribution of I.
   b. Compute $\mathrm{P}(I > 1)$.
   c. Compute $\mathrm{Cov}(I, B_1)$ and the correlation between the gains and the price change over the year.

3. Let X_t follow $\mathrm{d}X_t = -\frac{1}{2}\kappa X_t\,\mathrm{d}t + \frac{1}{2}\sigma\,\mathrm{d}B_t$, with constants $\kappa > 0$ and $\sigma > 0$, and model the short-term interest rate as $r_t = X_t^2$.

   a. Use Itô's formula to show that $\mathrm{d}r_t = \kappa(\theta - r_t)\,\mathrm{d}t + \sigma X_t\,\mathrm{d}B_t$, and find θ in terms of κ and σ.
   b. Show that $(\mathrm{d}r_t)^2 = \sigma^2 r_t\,\mathrm{d}t$.
   c. With $\kappa = 0.5$ and $\sigma = 0.2$, compute θ. When $r_t = 4\%$, compute the drift of the interest rate and the instantaneous standard deviation of $\mathrm{d}r_t$ per unit of $\sqrt{\mathrm{d}t}$.

4. Itô's formula is also a tool for computing moments and finding martingales.

   a. Apply Itô's formula to B_t^4, take expectations, and show that $\mathrm{E}(B_t^4) = 3t^2$. Repeat the argument with B_t^6 to compute $\mathrm{E}(B_t^6)$.
   b. Show that $M_t = e^{t/2}\cos(B_t)$ has zero drift, and use this to compute $\mathrm{E}(\cos(B_t))$.