Class 9

Itô's Formula: A Reminder

For an Itô process $\mathrm{d}X_t = a_t\,\mathrm{d}t + b_t\,\mathrm{d}B_t$ and a smooth function $f(x,t)$, Itô's formula reads

$$\mathrm{d}f(X_t,t) = \frac{\partial f}{\partial t}\,\mathrm{d}t + \frac{\partial f}{\partial x}\,\mathrm{d}X_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}X_t)^2, \qquad (\mathrm{d}X_t)^2 = b_t^2\,\mathrm{d}t. \tag{1}$$

Today we extend it to several processes driven by different, possibly correlated, sources of risk, and then see how changing the probability measure changes the drift of a Brownian motion.

Correlated Brownian Motions

Asset pricing models usually involve several sources of risk, such as a stock price and an interest rate. If B_t and W_t are independent Brownian motions, the sum $\sum_j \Delta B_j \Delta W_j$ has mean zero and variance $\sum_j \Delta t_j^2 \le \|\Pi\| t$, so it vanishes in the limit and $(\mathrm{d}B_t)(\mathrm{d}W_t) = 0$. For $-1 \le \rho \le 1$,

$$B_t^\rho = \rho B_t + \sqrt{1-\rho^2}\,W_t \tag{2}$$

is again a Brownian motion: its increments are independent and normal with variance $(\rho^2 + 1 - \rho^2)(t-s) = t-s$. Its increments have correlation ρ with those of B_t, and

$$(\mathrm{d}B_t)(\mathrm{d}B_t^\rho) = \rho(\mathrm{d}B_t)^2 + \sqrt{1-\rho^2}\,(\mathrm{d}B_t)(\mathrm{d}W_t) = \rho\,\mathrm{d}t.$$

Figure 1 builds B^ρ from the same two paths for $\rho = 0.8$ and $\rho = -0.8$.

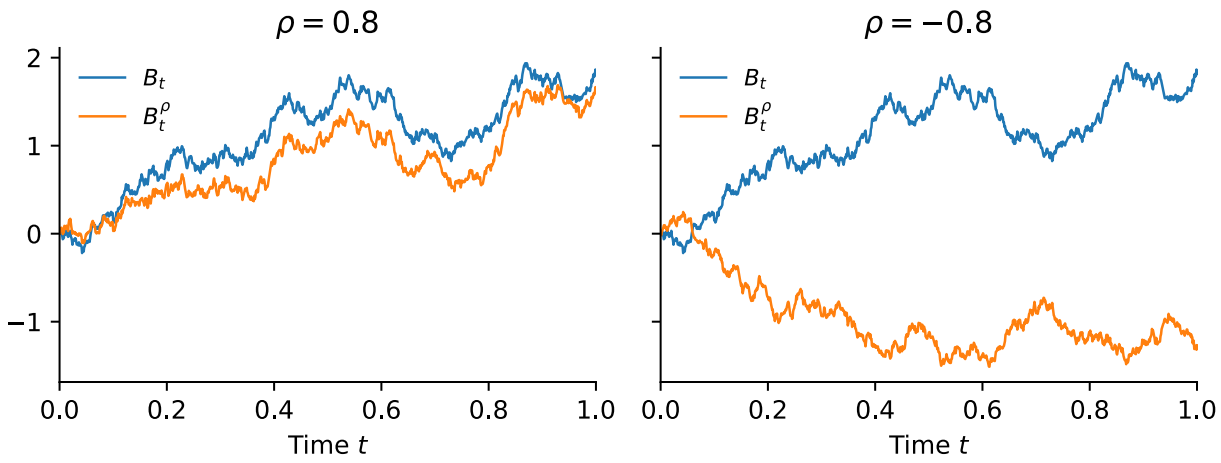


Figure 1: A Brownian path B_t and the Brownian motion $B_t^\rho = \rho B_t + \sqrt{1-\rho^2}\, W_t$ built from it and an independent path W_t. With $\rho = 0.8$ the two paths move together, and with $\rho = -0.8$ the second path mirrors the first.

Writing B_1 and B_2 for two Brownian motions whose increments have correlation ρ, the multiplication table is:

Product	$\mathrm{d}t$	$\mathrm{d}B_1$	$\mathrm{d}B_2$
$\mathrm{d}t$	0	0	0
$\mathrm{d}B_1$	0	$\mathrm{d}t$	$\rho\,\mathrm{d}t$
$\mathrm{d}B_2$	0	$\rho\,\mathrm{d}t$	$\mathrm{d}t$

For $\mathrm{d}X_t = a_t\,\mathrm{d}t + b_t\,\mathrm{d}B_1$ and $\mathrm{d}Y_t = c_t\,\mathrm{d}t + d_t\,\mathrm{d}B_2$, the table gives the **instantaneous covariance**

$$(\mathrm{d}X_t)(\mathrm{d}Y_t) = \rho\, b_t d_t\,\mathrm{d}t, \tag{3}$$

so the instantaneous correlation of $\mathrm{d}X_t$ and $\mathrm{d}Y_t$ is ρ when b_t and d_t are positive.

Itô's Formula for Two Processes

For a smooth function $f(x, y, t)$, the second-order expansion picks up a cross term:

$$\mathrm{d}f = \frac{\partial f}{\partial t}\,\mathrm{d}t + \frac{\partial f}{\partial x}\,\mathrm{d}X + \frac{\partial f}{\partial y}\,\mathrm{d}Y + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(\mathrm{d}X)^2 + \frac{\partial^2 f}{\partial x \partial y}(\mathrm{d}X)(\mathrm{d}Y) + \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(\mathrm{d}Y)^2. \tag{4}$$

With $f(x, y) = xy$ it gives the **product rule**, and for positive processes its version in percentage changes:

$$\mathrm{d}(X_t Y_t) = X_t\,\mathrm{d}Y_t + Y_t\,\mathrm{d}X_t + (\mathrm{d}X_t)(\mathrm{d}Y_t), \qquad \frac{\mathrm{d}(X_t Y_t)}{X_t Y_t} = \frac{\mathrm{d}X_t}{X_t} + \frac{\mathrm{d}Y_t}{Y_t} + \left(\frac{\mathrm{d}X_t}{X_t}\right)\left(\frac{\mathrm{d}Y_t}{Y_t}\right). \quad (5)$$

The correction vanishes when either process has no diffusion term. For a stock with $\mathrm{d}S_t/S_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t$ and a money-market account $\beta_t = e^{rt}$, the discount factor $1/\beta_t$ has no diffusion, so

$$\frac{\mathrm{d}(S_t/\beta_t)}{S_t/\beta_t} = (\mu - r)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t. \quad (6)$$

The discounted stock has drift equal to the expected excess return and the same volatility as the stock.

Girsanov's Theorem

To make discounted prices martingales we want to remove the drift in (6) without touching the paths. For a constant θ and horizon T, the exponential martingale $Z_t = \exp\left(-\theta B_t - \frac{1}{2}\theta^2 t\right)$ is positive with $\mathrm{E}(Z_T) = 1$, so it defines a new probability measure

$$\mathrm{P}^*(A) = \mathrm{E}\left(Z_T \mathbf{1}_{\{A\}}\right), \qquad \mathrm{E}^*(Y) = \mathrm{E}(Z_T Y). \quad (7)$$

Girsanov's theorem states that under P^*, $B_t^* = B_t + \theta t$ is a Brownian motion for $0 \le t \le T$. The same holds for an adapted θ_t with $B_t^* = B_t + \int_0^t \theta_s\,\mathrm{d}s$, as long as the corresponding Z_t is a martingale, for instance when θ_t is bounded. When $\theta > 0$, paths that end high get less weight, which shifts the distribution of B_t down, as in Figure 2.

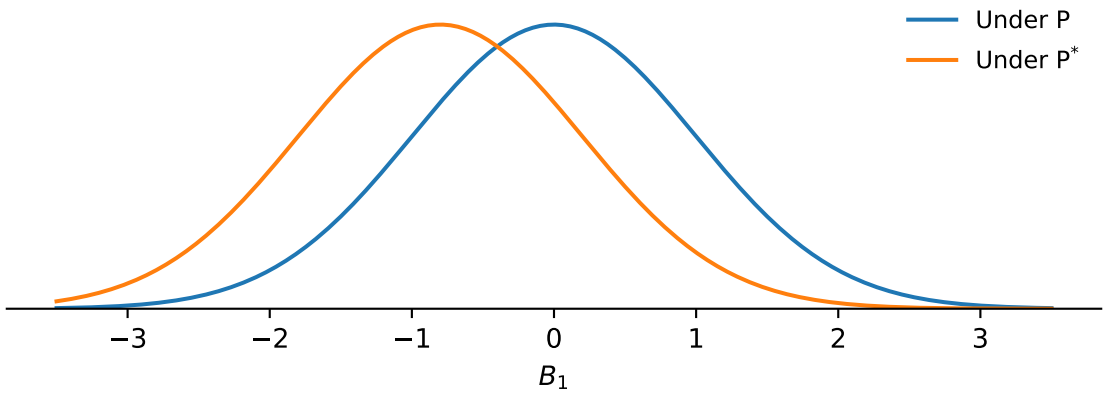


Figure 2: Density of B_1 under P and under P^* for $\theta = 0.8$. Reweighting each path by $Z_1 = e^{-\theta B_1 - \theta^2/2}$ shifts the normal density to the left by θ and leaves its spread unchanged.

In differential form, $\mathrm{d}B_t = \mathrm{d}B_t^* - \theta\,\mathrm{d}t$, so an Itô process becomes

$$\mathrm{d}X_t = (a_t - \theta b_t)\,\mathrm{d}t + b_t\,\mathrm{d}B_t^* \tag{8}$$

under P^*. The drift shifts by $-\theta b_t$ and the volatility is unchanged: quadratic variation is computed path by path, and P^* reweights paths without changing them. Since $Z_T > 0$, the two measures agree on which events are impossible. Choosing $\theta = (\mu - r)/\sigma$, the instantaneous Sharpe ratio, removes the drift in (6), and the stock follows $\mathrm{d}S_t/S_t = r\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^*$ under P^*. This is the construction behind the risk-neutral measure.

Practice Problems

1. A US investor holds a European stock. Its price in euros follows $\mathrm{d}S_t/S_t = \mu_S\,\mathrm{d}t + \sigma_S\,\mathrm{d}B_1$, and the exchange rate in dollars per euro follows $\mathrm{d}X_t/X_t = \mu_X\,\mathrm{d}t + \sigma_X\,\mathrm{d}B_2$, where the increments of B_1 and B_2 have correlation ρ. The dollar price of the stock is $D_t = X_t S_t$.

   a. Use the product rule to find $\mathrm{d}D_t/D_t$.
   b. With $\mu_S = 8\%$, $\sigma_S = 25\%$, $\mu_X = 1\%$, $\sigma_X = 10\%$ and $\rho = -0.3$, compute the drift and the volatility of the dollar return.

2. Let B_t and W_t be independent Brownian motions, and define $X_t = 0.6B_t + 0.8W_t$ and $Y_t = 0.6B_t - 0.8W_t$.

   a. Compute $(\mathrm{d}B_t)(\mathrm{d}X_t)$ and $(\mathrm{d}X_t)(\mathrm{d}Y_t)$.
   b. Use the product rule to find the drift of $B_t X_t$, and compute $\mathrm{E}(B_t X_t)$.
   c. Express W_t in terms of B_t and X_t.

3. A portfolio manager's position, in millions of dollars, follows $\mathrm{d}V_t = \mu_V\,\mathrm{d}t + 3\,\mathrm{d}B_1$, and the price of a stock index futures contract follows $\mathrm{d}F_t = \mu_F\,\mathrm{d}t + 2\,\mathrm{d}B_2$, where the increments of B_1 and B_2 have correlation $\rho = 0.8$. The manager sells h units of the futures, so the hedged position is $H_t = V_t - hF_t$.

   a. Compute $(\mathrm{d}H_t)^2$ as a function of h.
   b. Find the hedge ratio h that minimizes the instantaneous variance of the hedged position.
   c. Compute the instantaneous standard deviation of the hedged position at that h, and the fraction of the original variance the hedge removes.

4. The value of a firm's assets follows $\mathrm{d}V_t/V_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t$ with $\mu = 10\%$ and $\sigma = 20\%$, and the risk-free rate is $r = 4\%$.

   a. Find the θ that makes V_t/β_t have zero drift under P^*, and write the process followed by V_t under P^*.
   b. Find the distribution of $\ln(V_1/V_0)$ under P and under P^*.
   c. Compute the probability that the assets lose more than 25% of their value by time 1 under P and under P^*. Why is one larger?