Class 12

The Risk-Neutral Measure

A stock that pays no dividends over $[0, T]$ and a money-market account follow

$$\frac{\mathrm{d}S}{S} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B, \qquad \frac{\mathrm{d}\beta}{\beta} = r\,\mathrm{d}t, \qquad \frac{\mathrm{d}\Lambda}{\Lambda} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B,$$

where only the part of the SDF Λ correlated with the stock matters for pricing it. The pricing equation makes $\lambda = (\mu - r)/\sigma$ the stock's Sharpe ratio. The process $\mathcal{E} = \Lambda\beta$ satisfies $\mathrm{d}\mathcal{E}/\mathcal{E} = -\lambda\,\mathrm{d}B$, so it is a positive (local) martingale, and when it is a true martingale it defines the **risk-neutral measure** $\mathrm{dP}^*/\mathrm{d\,P} = \mathcal{E}_T/\mathcal{E}_0$. By Girsanov's theorem, $\mathrm{d}B^* = \mathrm{d}B + \lambda\,\mathrm{d}t$ is a P^*-Brownian motion, and

$$\frac{\mathrm{d}S}{S} = r\,\mathrm{d}t + \sigma\,\mathrm{d}B^*. \tag{1}$$

Every asset V driven by the same B has the same Sharpe ratio, so $\mathrm{d}V/V = r\,\mathrm{d}t + \sigma_V\,\mathrm{d}B^*$: under P^* all assets earn the risk-free rate. Since ΛV is a martingale,

$$V_0 = \mathrm{E}\left(\frac{\Lambda_T}{\Lambda_0}V_T\right) = \mathrm{E}^*\left(e^{-\int_0^T r_s\,\mathrm{d}s}V_T\right). \tag{2}$$

Prices are expected payoffs discounted at the risk-free rate, provided the expectation uses P^*.

Bonds, Futures, and Forwards

A zero-coupon bond has $V_T = 1$, so $P(T) = \mathrm{E}^*(e^{-\int_0^T r_s\,\mathrm{d}s})$. A futures position costs nothing to enter and pays its gains $\mathrm{d}F$ continuously, so $\mathrm{E}(\Lambda\,\mathrm{d}F) = 0$, or $\mathrm{E}^*(\,\mathrm{d}F) = 0$: the futures price is a P^*-martingale, and $F(T) = \mathrm{E}^*(S_T)$. The forward price $\varphi(T)$ gives the forward contract zero value, which yields

$$\varphi(T) = F(T) + \frac{\mathrm{Cov}^*\left(e^{-\int_0^T r_s\,\mathrm{d}s}, S_T\right)}{P(T)}. \tag{3}$$

Forward and futures prices coincide when this covariance is zero, in particular when r is constant.

The Black-Scholes Formula

With constant parameters, (1) gives $S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma B_T^*}$, so $\ln S_T$ is normal under P^* with mean $\ln S_0 + (r - \frac{1}{2}\sigma^2)T$ and variance $\sigma^2 T$. Splitting the call payoff into $S_T \mathbf{1}_{\{S_T>K\}} - K\mathbf{1}_{\{S_T>K\}}$,

$$C_0 = S_0 \mathrm{P}^S(S_T > K) - Ke^{-rT}\mathrm{P}^*(S_T > K), \tag{4}$$

where P^S is the measure with density $\Lambda_T S_T/(\Lambda_0 S_0)$, which uses the stock as the unit of account (a **change of numeraire**). Under P^S the stock has expected return $r + \sigma^2$, so only the drift differs across the three measures, as Figure 1 shows. Evaluating both probabilities,

$$C = S_0 N(d_1) - Ke^{-rT}N(d_2), \qquad P = Ke^{-rT}N(-d_2) - S_0 N(-d_1), \tag{5}$$

with

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T},$$

and $N(d)$ the standard normal CDF. Here $N(d_2)$ is the risk-neutral probability that the call is exercised. The expected return μ never appears. Subtracting the two prices gives put-call parity, $C - P = S_0 - Ke^{-rT}$, which holds in any model.

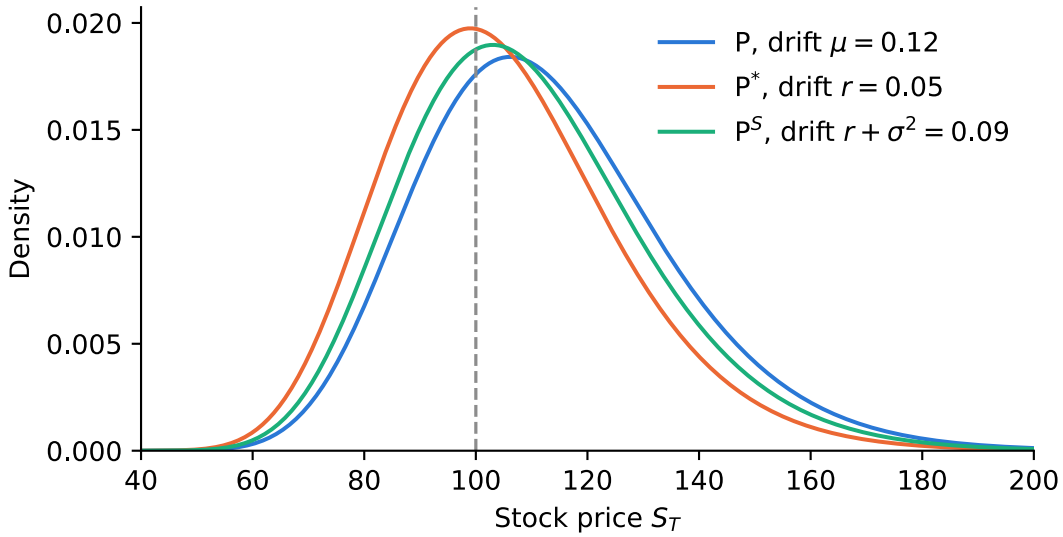


Figure 1: Density of the stock price S_T after one year under the three measures, with $S_0 = 100$, $\sigma = 0.20$, $\mu = 0.12$, and $r = 0.05$. The volatility is the same under all three; only the drift changes. The dashed line marks a strike of 100.

The Black-Scholes PDE

If $V(S,t)$ is the price of a derivative paying $f(S_T)$, Itô's lemma under (1) gives its drift, and that drift must equal rV:

$$rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} = rV, \qquad V(S,T) = f(S). \tag{6}$$

All derivatives on S solve the same equation and differ only in the terminal condition. The diffusion term $\sigma S\, \partial V/\partial S$ shows that holding the derivative and selling $\partial V/\partial S$ shares (its **delta**) removes all exposure to $\mathrm{d}B$, which was the original argument of Black and Scholes. The solution of (6) is the expectation (2), the Feynman-Kac formula.

The Merton Model

Real prices gap. Let N be a Poisson process with intensity λ (from here on, λ is a jump intensity) and Y_i iid $\mathcal{N}(m, s^2)$ log jump sizes, independent of B. In the model of Merton,

$$S_t = S_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2 - \lambda k\right)t + \sigma B_t + \sum_{i=1}^{N_t} Y_i\right), \qquad k = \mathrm{E}(e^Y) - 1 = e^{m+\frac{1}{2}s^2} - 1. \tag{7}$$

Each jump multiplies the price by e^{Y_i}, and k is the expected percentage jump. The $-\lambda k$ correction keeps $\mathrm{E}(S_t) = S_0 e^{\mu t}$. Given $N_t = n$, the log return is normal, so its density is a Poisson mixture of normals. Every cumulant beyond the variance comes from the jumps:

$$\kappa_2 = \sigma^2 t + \lambda t(m^2 + s^2), \qquad \kappa_3 = \lambda t\big(m^3 + 3ms^2\big), \qquad \kappa_4 = \lambda t\big(m^4 + 6m^2 s^2 + 3s^4\big). \tag{8}$$

With $m < 0$ the skewness $\kappa_3/\kappa_2^{3/2}$ is negative and the excess kurtosis κ_4/κ_2^2 is positive. They shrink like $1/\sqrt{t}$ and $1/t$, so jumps matter most over short horizons.

Itô's Formula with Jumps and Changes of Intensity

Compensated jumps. If the Z_i are iid and independent of N, then $\sum_{i \le N_t} Z_i - \lambda\,\mathrm{E}(Z)t$ is a martingale. A process with jumps is a martingale when its continuous drift plus λ times its expected jump is zero.

Itô's formula. For $X_t = X_0 + \int a_s\,\mathrm{d}s + \int b_s\,\mathrm{d}B_s + \sum_{i \le N_t} Z_i$ and f twice continuously differentiable,

$$\mathrm{d}f(X_t) = \left(a f'(X_t) + \frac{1}{2} b^2 f''(X_t)\right)\mathrm{d}t + b f'(X_t)\,\mathrm{d}B_t + \big(f(X_{t-} + \Delta X_t) - f(X_{t-})\big). \tag{9}$$

The jump term is the exact change in f, not a Taylor expansion. Applied to (7), $\mathrm{d}S_t/S_{t-} = (\mu - \lambda k)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t + (e^Y - 1)\,\mathrm{d}N_t$, whose jump term has compensator $\lambda k\,\mathrm{d}t$.

Changing the intensity. For $\lambda^* > 0$, the positive martingale

$$Z_t = \left(\frac{\lambda^*}{\lambda}\right)^{N_t} e^{-(\lambda^* - \lambda)t} \tag{10}$$

defines a measure under which N is Poisson with intensity λ^*. Brownian reweighting only shifts a drift, but the quadratic variation of N is random, so reweighting paths can change how often jumps occur. Jump sizes can be reweighted too.

Credit Spreads

A firm defaults at the first event of N, and bondholders lose a fraction L at default. If the SDF jumps by the factor $1 + \eta$ at default, $\Lambda_t = e^{-rt}(1+\eta)^{N_t} e^{-\lambda\eta t}$, then $\Lambda\beta$ is the density (10) with $\lambda^* = \lambda(1+\eta)$, and a bond priced $e^{-c(T-t)}$ before default has spread

$$c - r = \lambda^* L = (1 + \eta)\lambda L. \tag{11}$$

With $\eta > 0$, defaults hit in bad times, and the spread exceeds the expected loss rate λL.

Option Pricing with Jumps

A stock position hedges $\mathrm{d}B$ but not a jump of random size, because an option's value changes nonlinearly in the jump. The market is **incomplete**: many risk-neutral measures price the stock, each with $\mathrm{d}S_t/S_{t-} = (r - \lambda^* k^*)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t^* + (e^Y - 1)\,\mathrm{d}N_t$. Merton assumed that jump risk is diversifiable and unpriced, so λ and the jump distribution are the same under P and P^*. The call price is then a Poisson mixture of Black-Scholes prices,

$$C = \sum_{n=0}^{\infty} e^{-\lambda' T} \frac{(\lambda' T)^n}{n!} C^{BS}(S_0, K, r_n, \sigma_n, T), \tag{12}$$

with $\lambda' = \lambda(1 + k)$, $\sigma_n^2 = \sigma^2 + ns^2/T$, and $r_n = r - \lambda k + n \ln(1 + k)/T$. Downward jumps raise the implied volatility of low strikes, producing the **skew** of Figure 2, which is steepest at short maturities.

For models without a formula like (12), (4) still holds, and both probabilities come from the risk-neutral characteristic function ϕ of $\ln S_T$ by the inversion formula $\mathrm{P}(X > x) = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Im}(e^{-iux}\phi(u))/u\,\mathrm{d}u$, using $\phi^S(u) = \phi(u - i)/\phi(-i)$ for P^S.

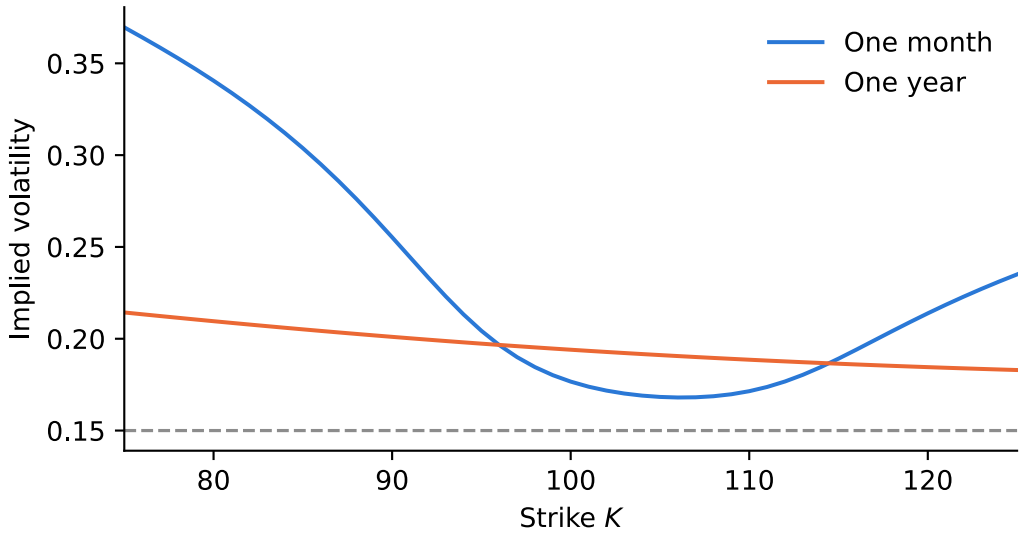


Figure 2: Black-Scholes implied volatility of Merton call prices against the strike for one-month and one-year options, with $S_0 = 100$, $r = 0.05$, $\sigma = 0.15$, $\lambda = 1$, $m = -0.08$, and $s = 0.10$. The dashed line is the diffusion volatility.

Practice Problems

1. A stock follows $\mathrm{d}S/S = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$ with $\mu = 0.10$ and $\sigma = 0.20$, its price today is $S_0 = 100$, and the risk-free rate is a constant $r = 0.04$.

   a. Compute λ and write the dynamics of S under P^*.
   b. Another asset is driven by the same Brownian motion and has volatility $\sigma_V = 0.50$. What is its expected return under P? Under P^*?
   c. Compute the two-year futures price and the two-year forward price. Why are they equal?
   d. A *log contract* pays $\ln S_T$ at $T = 2$. Use (2) to compute its price today.

2. A pension fund holds a stock index at $S_0 = 100$ and buys a one-year European put with strike $K = 95$ as insurance. The index volatility is $\sigma = 0.25$ and $r = 0.04$. Use $N(0.49) = 0.6879$ and $N(0.24) = 0.5948$.

   a. Compute d_1 and d_2, and the risk-neutral probability that the put is exercised.
   b. The put's payoff is a long position in a contract paying K if $S_T < K$ and a short position in a contract paying S_T if $S_T < K$. Price each contract using (4) and its analog for the put, and compute the put price.
   c. Use put-call parity to price the call with the same strike, and check it with (5).

3. A firm defaults with historical intensity $\lambda = 0.02$ per year, bondholders lose $L = 0.5$ at default, and $r = 0.04$. Its bonds trade at a credit spread of 300 basis points.

   a. Compute the expected loss rate, the risk-neutral intensity λ^*, and the jump η in the SDF at default.
   b. Compute the price of a five-year zero-coupon bond with face value 1 before default, and compare it with a risk-free bond.
   c. Compute the probability that the firm defaults within five years under P and under P^*.
   d. Evaluate the density Z_5 in (10) on paths with no default and on paths with exactly one default by year five. Explain why default paths receive more weight under P^*.