Stochastic Foundations for Finance — Prof. Lorenzo Naranjo
FIN 5380 — Fall 2026

Problem Set 1

Instructions: This problem set is due on 10/29 at 11:59 pm Central Time and is an individual assignment. All problems must be handwritten. Write full sentences to answer questions that require an explanation. Scan your work and submit a PDF file.

Problem 1. The economy next year can be in a recession, a normal state, or a boom, with probabilities 0.25, 0.50, and 0.25. The table below gives the returns, in percent, of a stock A and a bond B in each state.

State	Probability	R_A (%)	R_B (%)
Recession	0.25	-10	6
Normal	0.50	8	4
Boom	0.25	20	2

a. Compute $\mathrm{E}(R_A)$, $\mathrm{E}(R_B)$, $\mathrm{SD}(R_A)$, and $\mathrm{SD}(R_B)$.
b. Compute $\mathrm{Cov}(R_A, R_B)$ and the correlation $\rho_{A,B}$.
c. A portfolio invests a fraction w of its wealth in the stock and $1-w$ in the bond, so that its return is $R_p = wR_A + (1-w)R_B$. Compute the expected return and the standard deviation of the portfolio when $w = 0.5$.

Problem 2. Let X be equal to -1 if the stock market falls next year and $+1$ if it rises, and let Y be the number of defaults next year in a portfolio of two corporate bonds. The joint probability mass function of X and Y is given below.

$X \setminus Y$	0	1	2
-1	0.10	0.15	0.15
1	0.45	0.10	0.05

a. Find the marginal probability mass functions $p_X(x)$ and $p_Y(y)$.
b. Compute $\mathrm{E}(Y)$ and $\mathrm{V}(Y)$.

c. Compute $\mathrm{E}(Y \mid X = -1)$ and $\mathrm{E}(Y \mid X = 1)$, and verify the law of iterated expectations $\mathrm{E}(\mathrm{E}(Y \mid X)) = \mathrm{E}(Y)$.
d. Compute $\mathrm{Cov}(X, Y)$. Are X and Y independent? Explain.
e. Let U take the values -2, -1, 1, and 2, each with probability $1/4$, and let $V = U^2$. Show that $\mathrm{Cov}(U, V) = 0$ but that U and V are not independent.

Problem 3. Let X_1 and X_2 be independent random variables, each equal to $+1$ with probability $p = 0.6$ and to -1 with probability $q = 0.4$. Define $S_0 = 0$, $S_1 = X_1$, and $S_2 = X_1 + X_2$. Let $\mathcal{F}_1$ be the information set generated by X_1 and $\mathcal{F}_2$ the information set generated by X_1 and X_2.

a. List the four outcomes of the sample space Ω and their probabilities, together with the values of S_1 and S_2 for each outcome.
b. Write down the partitions of Ω that generate $\mathcal{F}_1$ and $\mathcal{F}_2$, and list all the events in $\mathcal{F}_1$.
c. Is S_2 known given $\mathcal{F}_1$? Is S_1 known given $\mathcal{F}_2$? Explain.
d. Compute $\mathrm{E}(S_2 \mid \mathcal{F}_1)$ on each cell of the partition that generates $\mathcal{F}_1$, and verify that $\mathrm{E}(\mathrm{E}(S_2 \mid \mathcal{F}_1)) = \mathrm{E}(S_2)$.
e. Let $M_n = S_n - n(p - q)$. Use the properties of conditional expectation to show that $\mathrm{E}(M_2 \mid \mathcal{F}_1) = M_1$.

Problem 4. A firm defaults at a random time T, measured in years, that has an exponential distribution with intensity $\lambda = 0.04$ per year.

a. Compute the probability that the firm defaults within five years, and the median time to default.
b. Compute $\mathrm{P}(T > 10 \mid T > 5)$ and compare it with $\mathrm{P}(T > 5)$. Show that, for any $s, t \geq 0$, $\mathrm{P}(T > s + t \mid T > s) = \mathrm{P}(T > t)$, and explain in words what this property says about a firm that has survived s years.
c. A computer generates the uniform draws $u = 0.20$, 0.65, and 0.90. Use the inverse transform method to turn them into three simulated default times.

Problem 5. Suppose that X is a normally distributed random variable with mean $\mu = 12$ and standard deviation $\sigma = 20$.

a. What is the probability that $X \leq 0$?
b. What is the probability that $4 < X \leq 10$?
c. Compute the 90%, 95%, and 99% probability intervals of X.

d. Let $Z \sim \mathcal{N}(0,1)$. Compute $\mathrm{P}(Z^2 \leq 1)$ and $\mathrm{P}(Z^2 > 3.8416)$.

Problem 6. The price Y of a stock one year from now is lognormally distributed, with $\ln(Y) \sim \mathcal{N}(\mu, \sigma^2)$, $\mu = 4.6$, and $\sigma = 0.25$.

a. What is the probability that $Y \leq 80$?
b. What is the probability that $90 < Y \leq 110$?
c. What is the probability that $Y \leq 0$?
d. Compute the median, the expected value, and the standard deviation of Y. Explain why the expected value is larger than the median.
e. Compute the 95% probability interval of Y. Is it symmetric around the median?

Problem 7. The annual returns of a stock, R_A, and of the market portfolio, R_M, have a bivariate normal distribution with $\mathrm{E}(R_A) = 0.10$, $\mathrm{SD}(R_A) = 0.30$, $\mathrm{E}(R_M) = 0.07$, $\mathrm{SD}(R_M) = 0.15$, and correlation $\rho = 0.6$.

a. Compute $\mathrm{Cov}(R_A, R_M)$.
b. A portfolio invests half of its wealth in the stock and half in the market. Compute the expected return and the standard deviation of the portfolio, and the probability that the portfolio loses money over the year.
c. Suppose we learn that the market returned -20% this year. Compute the conditional expected return and the conditional standard deviation of the stock, and the conditional probability that the stock loses money. Compare it with the unconditional probability that the stock loses money.
d. Compute the slope $\rho\,\mathrm{SD}(R_A)/\mathrm{SD}(R_M)$, the beta of the stock with respect to the market, and use it to explain your answer to the previous part.

Normal Distribution Table

The table below computes $\Phi(z) = \mathrm{P}(Z \leq z)$ if $Z \sim \mathcal{N}(0,1)$ and $z \geq 0$. The rows denote the first decimal whereas the columns denote the second decimal. If you need $\Phi(-z)$, you can always use $\Phi(-z) = 1 - \Phi(z)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990