Problem Set 2

Instructions: This problem set is due on 11/5 at 11:59 pm Central Time and is an individual assignment. All problems must be handwritten. Write full sentences to answer questions that require an explanation. Scan your work and submit a PDF file.

Problem 1. The number of defaults X next year in a small portfolio of loans has the probability mass function $p_X(0) = 0.5$, $p_X(1) = 0.3$, and $p_X(2) = 0.2$.

a. Write down the probability generating function $G_X(z)$.
b. Use G_X to compute $\mathrm{E}(X)$ and $\mathrm{V}(X)$.
c. A second portfolio has a number of defaults Y that is independent of X and has the same distribution. Compute the probability generating function of $X + Y$ and read off the probability mass function of the total number of defaults.
d. Use your answer to the previous part to compute $\mathrm{P}(X + Y \geq 3)$.

Problem 2. A trading strategy makes money on each day with probability $p = 0.6$, independently across days. Let W be the number of days needed to obtain three winning days.

a. Explain why $W = X_1 + X_2 + X_3$, where X_1, X_2, and X_3 are independent random variables with the geometric distribution of parameter p.
b. Find the probability generating function of W.
c. Use the probability generating function to compute $\mathrm{E}(W)$ and $\mathrm{V}(W)$.
d. Compute $\mathrm{P}(W = 3)$ and $\mathrm{P}(W = 4)$.

Problem 3. The number of orders N that arrive at a market maker during a minute has a Poisson distribution with parameter $\lambda = 20$. Each order is independently for one lot with probability $1/2$ and for two lots with probability $1/2$, and is independent of N. Let S be the total number of lots ordered during the minute.

a. Find the probability generating function of S.
b. Use it to compute $\mathrm{E}(S)$ and $\mathrm{V}(S)$. Compare $\mathrm{V}(S)$ with $\mathrm{E}(S)$.

c. A bank holds $n = 1{,}000$ independent loans, each with a default probability of 0.002 over the next year. Compute the exact probability of no defaults, and the probability of no defaults under the Poisson approximation.
d. Compute the probability of at most two defaults, exactly and with the Poisson approximation.

Problem 4. Let $Y = e^X$ where $X = \mu + \sigma Z$, $\mu = 1$, $\sigma = 2$, and $Z \sim \mathcal{N}(0, 1)$. In addition, the daily log returns $R_1, R_2, \ldots, R_{250}$ of a stock are independent and normally distributed with mean 0.0004 and standard deviation 0.01, and the annual log return is $R = R_1 + \cdots + R_{250}$.

a. Use the moment generating function of X to compute $\mathrm{E}(Y^{0.5})$ and $\mathrm{E}(Y^{-1})$.
b. Use moment generating functions to find the distribution of R.
c. Compute the probability that the stock loses money over the year, that is, $\mathrm{P}(R < 0)$.
d. Compute the expected gross return $\mathrm{E}(e^R)$ over the year.

Problem 5. Let $X_1, X_2, \ldots$ be independent random variables, each equal to $+1$ with probability p and to -1 with probability $q = 1 - p$, and let $S_n = X_1 + \cdots + X_n$ with $S_0 = 0$.

a. Compute the moment generating function $M_X(\theta)$ and the cumulant generating function $K_X(\theta)$ of X_1, and use K_X to find the mean and the variance of X_1.
b. Find the value $\theta^* \neq 0$ such that $K_X(\theta^*) = 0$, and show that $e^{\theta^*} = q/p$.
c. Let $\mathcal{F}_n$ be the information set generated by $X_1, \ldots, X_n$. Use the properties of conditional expectation to show that $\mathrm{E}\left((q/p)^{S_{n+1}} \mid \mathcal{F}_n\right) = (q/p)^{S_n}$.
d. Now let $p = q = 1/2$. Show that the characteristic function of $S_n/\sqrt{n}$ is $\cos(u/\sqrt{n})^n$, and compare its value at $u = 1$ for $n = 10$ and $n = 100$ with the characteristic function of a standard normal random variable. Which theorem does this illustrate?

Problem 6. A gambler bets \$1 on each round of a game, wins each round with probability p, and loses it with probability $q = 1 - p$. The gambler starts with no winnings, $S_0 = 0$, and stops as soon as the net winnings reach A or $-B$.

a. Use the ruin probability for a biased coin derived in the notes to complete the table below, which reports the probability of winning \$100 before losing \$100.

Chance of winning one round	0.500	0.495	0.490	0.480	0.470
Chance to win $100					

b. Argue that, if your chance of winning each round is less than 0.5, it is much better to bet $100 at once than $1 at a time.

c. Let $g(K) = \mathrm{E}(\tau \mid S_0 = K)$ be the expected number of rounds until the game stops when the gambler starts at K. Explain why

$$g(K) = 1 + p\, g(K+1) + q\, g(K-1), \qquad g(-B) = g(A) = 0.$$

d. Show that $g(K) = K/(q-p)$ solves the equation in the previous part without the boundary conditions, and conclude that the general solution is

$$g(K) = \frac{K}{q-p} + \alpha + \beta\left(\frac{q}{p}\right)^K.$$

e. A bet on red in American roulette wins with probability $p = 18/38$. Compute the expected duration $g(0)$ of the game when $A = B = 10$, and compare it with the expected duration $AB = 100$ of a fair game.

Normal Distribution Table

The table below computes $\Phi(z) = \mathrm{P}(Z \leq z)$ if $Z \sim \mathcal{N}(0,1)$ and $z \geq 0$. The rows denote the first decimal whereas the columns denote the second decimal. If you need $\Phi(-z)$, you can always use $\Phi(-z) = 1 - \Phi(z)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990