Problem Set 3

Instructions: This problem set is due on 11/12 at 11:59 pm Central Time and is an individual assignment. All problems must be handwritten. Write full sentences to answer questions that require an explanation. Scan your work and submit a PDF file.

Problem 1. Let $X_1, X_2, \ldots$ be independent random variables, each equal to $+2$ with probability $1/3$ and to -1 with probability $2/3$. Let $S_0 = 0$, $S_n = X_1 + \cdots + X_n$, and let $\mathcal{F}_n$ be the information set generated by $X_1, \ldots, X_n$.

a. Show that $\{S_n\}$ is a martingale with respect to $\{\mathcal{F}_n\}$.
b. Find the constant σ^2 such that $\{S_n^2 - n\sigma^2\}$ is a martingale, and prove that it is.

Problem 2. A stock price starts at $P_0 = 100$ and each period is multiplied by 1.2 with probability p or by 0.9 with probability $1 - p$, independently across periods. Let $\mathcal{F}_n$ be the information generated by the first n price moves.

a. Find the probability p that makes $\{P_n\}$ a martingale.
b. A bank account pays an interest rate of $r = 2\%$ per period. Find the probability p that makes the discounted price $\{P_n/(1.02)^n\}$ a martingale.
c. With the probability you found in the previous part, compute $\mathrm{E}(P_3 \mid \mathcal{F}_1)$ when the price went up in the first period.

Problem 3. A gambler bets on a fair coin, winning or losing the stake on each round, so that the gain from a bet of one dollar on round k is $X_k = \pm 1$ with probability $1/2$ each.

a. The gambler stakes one dollar on round k if round $k-1$ was lost and two dollars if it was won, and one dollar on the first round. Explain why this strategy is predictable, and compute the expected wealth after n rounds.
b. A second gambler stakes $H_k = X_k$ dollars on round k. Compute the wealth after n rounds and explain why this strategy is not allowed.
c. A third gambler follows the doubling strategy, starting with a stake of one dollar, but has a line of credit of only \$63, so that the gambler must stop after six consecutive losses.

Compute the probability that the gambler ends with a gain of one dollar, the probability and size of the loss, and the expected gain.

Problem 4. Let $\{S_n\}$ be a random walk with $S_0 = 0$ whose steps are $+1$ with probability p and -1 with probability $q = 1 - p$, and let $\mathcal{F}_n$ be the information generated by the first n steps.

a. Which of the following random times are stopping times? Explain. (i) The first time the walk reaches 5. (ii) One period after the first time the walk reaches 5. (iii) One period before the first time the walk reaches 5. (iv) The last time the walk is at zero before period 10.
b. A trader's account starts at zero and gains or loses one dollar each period, with $p = 18/38$. The trader stops after winning or losing \$10. Use the martingale $S_n - n(p - q)$ and the ruin probability for a biased coin to compute the expected number of periods until the trader stops, and compare it with your answer to Problem 6 of Problem Set 2.
c. Let $p = 0.55$, so that the walk drifts upward, and suppose there is no upper barrier. Use the martingale $(q/p)^{S_n}$ to argue that the probability of ever losing \$10 is $(q/p)^{10}$, and compute it. How large must the initial capital be for the probability of ever losing it all to be below 1%?

Problem 5. Insurance claims arrive as a Poisson process $\{N_t\}$ with rate $\lambda = 3$ per month.

a. Compute the probability of no claims during a month and the probability of at least two claims during two months.
b. Given that three claims arrived during the first two months, compute the probability that exactly two of them arrived during the first month.
c. Compute the expected time until the fifth claim, and the probability that the second claim arrives within one month.
d. For $s < t$, compute $\mathrm{E}(N_s N_t)$ and $\mathrm{Cov}(N_s, N_t)$. (Hint: write $N_t = N_s + (N_t - N_s)$.)
e. Show that $2^{N_t} e^{-\lambda t}$ is a martingale.

Problem 6. Defaults in two sectors of a loan portfolio arrive as independent Poisson processes with rates $\lambda_1 = 0.5$ and $\lambda_2 = 1.5$ per year.

a. Compute the probability of no defaults in the portfolio over the next year.
b. Show that the probability that the first default comes from sector 1 is $\lambda_1/(\lambda_1 + \lambda_2)$, and compute it. (Hint: the first default times $T^{(1)}$ and $T^{(2)}$ of the two sectors are independent exponential random variables. Compute $\mathrm{P}(T^{(1)} < T^{(2)})$ by conditioning on $T^{(1)}$.)

c. Each default causes a loss that is exponentially distributed with a mean of $10 million, independently of everything else. Compute the expected value and the standard deviation of the total loss over the next year.