Problem Set 4

Instructions: This problem set is due on 11/19 at 11:59 pm Central Time and is an individual assignment. All problems must be handwritten. Write full sentences to answer questions that require an explanation. Scan your work and submit a PDF file.

Problem 1. Let $\{B_t\}$ be a Brownian motion, and let $s = 0.7$ and $t = 2$.

a. Compute $\mathrm{P}(B_s \leq -0.5)$, $\mathrm{P}(B_t > 1)$, and $\mathrm{P}(|B_t| \leq 1.5)$.
b. Compute $\mathrm{P}(|B_t - B_s| > 2)$.
c. Compute the correlation between B_s and B_t.
d. Find the distribution of $B_s + B_t$, and compute $\mathrm{P}(B_s + B_t > 2)$. (Hint: write $B_s + B_t = 2B_s + (B_t - B_s)$.)

Problem 2. The logarithm of the value of a firm's assets, relative to their current value, follows a Brownian motion with drift $X_t = \mu t + \sigma B_t$, with $\mu = 0.05$ and $\sigma = 0.25$ per year.

a. The firm defaults if the value of its assets falls by 30%. Compute the probability that the firm ever defaults.
b. Compute the probability that the value of the assets falls by 30% before it rises by 50%.

Problem 3. The log price of a stock follows $X_t = \mu t + \sigma B_t$. An analyst estimates the volatility from n daily log returns over one year, $T = 1$, using the *realized variance*

$$Q_n = \sum_{j=0}^{n-1} (X_{t_{j+1}} - X_{t_j})^2, \qquad t_j = j/n.$$

a. Assume first that $\mu = 0$. Show that $\mathrm{E}(Q_n) = \sigma^2$ and $\mathrm{V}(Q_n) = 2\sigma^4/n$.
b. With $\sigma = 0.20$ and $n = 252$, compute the standard deviation of Q_n, and compare it with σ^2.
c. Now let $\mu \neq 0$. Show that the drift adds μ^2/n to $\mathrm{E}(Q_n)$, and explain why the realized variance measures the volatility precisely while the drift cannot be estimated from a year of data in the same way.

Problem 4. Let $\{B_t\}$ be a Brownian motion and $T > 0$.

a. Compute the mean and the variance of $I_T = \int_0^T s \, \mathrm{d}B_s$, and find its distribution.
b. Compute the mean and the variance of $J_T = \int_0^T B_s \, \mathrm{d}B_s$ using the properties of the Itô integral, and check your answer using $J_T = \frac{1}{2}(B_T^2 - T)$.
c. Apply Itô's formula to B_t^3, and use the result to write $\int_0^T B_s^2 \, \mathrm{d}B_s$ in terms of B_T and an ordinary integral of B_s.
d. Use Itô's formula to show that $B_t^3 - 3tB_t$ has zero drift, and conclude that it is a martingale.

Problem 5. Consider an *Ornstein-Uhlenbeck* (or mean-reverting) process for an interest rate y such that

$$\mathrm{d}y_t = \kappa(\theta - y_t)\,\mathrm{d}t + \sigma \, \mathrm{d}B_t,$$

where $\kappa > 0$, θ, and $\sigma > 0$ are constants.

a. Apply Itô's formula to $e^{\kappa t} y_t$ to show that

$$y_t = e^{-\kappa t} y_0 + \theta\left(1 - e^{-\kappa t}\right) + \sigma e^{-\kappa t} \int_0^t e^{\kappa s} \, \mathrm{d}B_s.$$

b. Compute $\mathrm{E}(y_t)$ and $\mathrm{V}(y_t)$, and explain why y_t is normally distributed.
c. Compute the limits of $\mathrm{E}(y_t)$ and $\mathrm{V}(y_t)$ as $t \to \infty$, and explain how they show that y_t reverts to θ. Can the interest rate become negative in this model?

Problem 6. Consider two assets whose price processes are given by

$$\frac{\mathrm{d}\mathbf{S}}{\mathbf{S}} = \boldsymbol{\mu}\,\mathrm{d}t + \boldsymbol{\sigma}\,\mathrm{d}\mathbf{B},$$

where $\mathrm{d}\mathbf{B}$ is a vector of three independent Brownian motions B_1, B_2, and B_3, and

$$\boldsymbol{\sigma} = \begin{pmatrix} 0.3 & -0.1 & 0.2 \\ 0.15 & 0.2 & -0.05 \end{pmatrix}.$$

a. Compute the instantaneous volatility of each asset and the instantaneous correlation between their returns.
b. Use the product rule to find the process followed by the ratio $Y = S_1/S_2$, the price of asset 1 measured in units of asset 2. (Hint: first find $\mathrm{d}(1/S_2)$ with Itô's formula.)

c. Find a Brownian motion $B_4 = a_1 B_1 + a_2 B_2 + a_3 B_3$ whose increments are independent of the returns of the two assets.

Problem 7. Let $\{B_t\}$ be a Brownian motion under P, let $\theta = 0.5$, and define the probability measure P^* by $\mathrm{P}^*(A) = \mathrm{E}(Z_1 \mathbf{1}_{\{A\}})$, where $Z_1 = e^{-\theta B_1 - \frac{1}{2}\theta^2}$.

a. Use Girsanov's theorem to find the distribution of B_1 under P^*, and compute $\mathrm{P}(B_1 > 0)$ and $\mathrm{P}^*(B_1 > 0)$.
b. Verify your answer for the mean by computing $\mathrm{E}^*(B_1) = \mathrm{E}(Z_1 B_1)$ directly. (Hint: differentiate the normal moment generating function $\mathrm{E}(e^{cB_1}) = e^{\frac{1}{2}c^2}$ with respect to c.)
c. A stock follows $\mathrm{d}S_t/S_t = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t$ with $\mu = 12\%$ and $\sigma = 20\%$, and the risk-free rate is $r = 4\%$. Find the value of θ that makes the discounted stock price a martingale under P^*, and write the process followed by S_t under P^*.

Normal Distribution Table

The table below computes $\Phi(z) = \mathrm{P}(Z \leq z)$ if $Z \sim \mathcal{N}(0,1)$ and $z \geq 0$. The rows denote the first decimal whereas the columns denote the second decimal. If you need $\Phi(-z)$, you can always use $\Phi(-z) = 1 - \Phi(z)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990