Problem Set 5

Instructions: This problem set is due on 12/3 at 11:59 pm Central Time and is an individual assignment. All problems must be handwritten. Write full sentences to answer questions that require an explanation. Scan your work and submit a PDF file.

Problem 1. A stock price follows a geometric Brownian motion

$$\frac{\mathrm{d}S_t}{S_t} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B_t,$$

with an expected return of $\mu = 12\%$ per year, a volatility of $\sigma = 35\%$ per year, and a current price of $S_0 = \$100$.

a. Compute the mean and the standard deviation of the log price $\ln S_T$ two years from now.
b. Compute the expected price and the median price two years from now, and explain why they differ.
c. Compute the probability that the price is below $100 two years from now, and the probability that it is above $120.
d. Compute the 95% probability interval for the price two years from now.

Problem 2. In this problem the stock price follows the geometric Brownian motion of the previous problem, with constant μ and σ, and the risk-free rate r is constant.

a. Show that $Y_t = S_t e^{-\mu t}$ is a martingale.
b. Let $X_t = \sqrt{S_t}$. Use Itô's formula to find the expected return and the volatility of X_t, and compare its expected return with $\frac{1}{2}\mu$.
c. A portfolio holds a fraction w of its value in the stock and $1 - w$ in a money-market account earning r, and rebalances continuously, so that its value W_t satisfies $\mathrm{d}W_t/W_t = w\ \mathrm{d}S_t/S_t + (1-w)r\,\mathrm{d}t$. Find the growth rate of $\ln W_t$, and the weight w that maximizes it.

Problem 3. The stochastic discount factor and a money-market account follow

$$\frac{\mathrm{d}\Lambda}{\Lambda} = -r\,\mathrm{d}t - \lambda\,\mathrm{d}B, \qquad \frac{\mathrm{d}\beta}{\beta} = r\,\mathrm{d}t,$$

where r and λ are constants.

a. Use the product rule to show that $\Lambda\beta$ has zero drift.
b. A stock that pays no dividends follows $\mathrm{d}S/S = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$. Find the value of λ such that ΛS has zero drift, and interpret it.
c. In a consumption-based model, the SDF is $\Lambda_t = e^{-\phi t}C_t^{-\gamma}$, with consumption growth $\mathrm{d}C/C = \mu_C\,\mathrm{d}t + \sigma_C\,\mathrm{d}B_C$. With $\phi = 0.01$, $\gamma = 2$, $\mu_C = 0.02$, and $\sigma_C = 0.02$, compute the risk-free rate and the risk premium of a stock with volatility $\sigma = 0.16$ whose return has a correlation of 0.5 with consumption growth.
d. Historically, the stock market has earned a risk premium of about 6% per year. What coefficient of relative risk aversion γ would the model in the previous part need to deliver this premium? Comment on your answer.

Problem 4. A stock trades at $S_0 = \$50$ and has a volatility of $\sigma = 30\%$ per year, and the risk-free rate is $r = 4\%$. Consider European options on the stock with strike $K = \$55$ and maturity $T = 0.5$ years. Round d_1 and d_2 to two decimals.

a. Compute d_1 and d_2, and the risk-neutral probability that the call expires in the money.
b. Compute the prices of the call and the put, and verify put-call parity.
c. A *digital call* pays \$1 at time T if $S_T > K$ and nothing otherwise. Compute its price.
d. Suppose the expected return of the stock is $\mu = 12\%$. Compute the probability under P that the call expires in the money, and explain why it differs from the risk-neutral probability. Does the call price depend on μ?

Problem 5. In the Black-Scholes model, the price $V(S,t)$ of any derivative written on the stock satisfies

$$rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} = rV,$$

with a terminal condition $V(S,T)$ given by the payoff.

a. Show that the value of a long forward contract with delivery price K, $V = S - Ke^{-r(T-t)}$, satisfies the equation. What is its terminal condition?

b. Show that the value of a zero-coupon bond paying one dollar at T, $V = e^{-r(T-t)}$, satisfies the equation. What is its terminal condition?
c. A contract pays S_T^2 at time T. Show that $V = S^2 e^{(r+\sigma^2)(T-t)}$ satisfies the equation and the terminal condition.
d. Show that the futures price $F_t = S_t e^{r(T-t)}$ has zero drift under the risk-neutral measure.

Normal Distribution Table

The table below computes $\Phi(z) = \mathrm{P}(Z \leq z)$ if $Z \sim \mathcal{N}(0,1)$ and $z \geq 0$. The rows denote the first decimal whereas the columns denote the second decimal. If you need $\Phi(-z)$, you can always use $\Phi(-z) = 1 - \Phi(z)$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990